

# BEAM DYNAMICS IN HIGH ENERGY PARTICLE ACCELERATORS

SECOND EDITION

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# BEAM DYNAMICS IN HIGH ENERGY PARTICLE ACCELERATORS

SECOND EDITION

**Andrzej Wolski**

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**Second Edition**

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To my parents, for all the many years of patience and support.

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## Preface to the Second Edition

Accelerator physics is an area of active research, and there have inevitably been many developments in the field since the first edition of this book was published, nearly a decade ago. Continuing improvements in technology and techniques mean that accelerators can now reach parameter regimes that were not accessible at that time: this is exemplified by the rapid progress in x-ray free-electron lasers, which enabled the growing field of ultra-fast science, and with ultra-low emittance storage rings that can now operate as diffraction-limited light sources. There has also been significant progress in recent years with plasma-wakefield acceleration techniques, which offer the prospect of achieving accelerating gradients orders of magnitude higher than those that can be achieved in conventional radio-frequency (RF) accelerating structures. However, control of the beam quality and stability in plasma-wakefield accelerators remains a considerable challenge, so facilities based on this technology for scientific, industrial or medical applications still seem some way off. In terms of beam dynamics, the principles underlying the operation of RF-based accelerators today are the same as they were ten (or more) years ago. Hence, while this second edition has offered the opportunity to add some selected, specific topics, very major changes to the scope have not been needed. Of course, the second edition has also allowed me to correct a number of regrettable errors in the first edition, to revise and improve the text in many places, and to update and add to the references.

New topics in this edition include an analytical treatment of multipole fringe fields (in Chapter 1), the measurement of normal mode emittances (Chapter 5), and the Störmer–Verlet method for numerical integration of conservative systems (Chapter 10). The inclusion of all these topics has been motivated not only by their relevance to my own research and teaching

activities in the past few years, but also by their potentially very widespread practical value in accelerator beam dynamics. A further topic that is of increasing interest and importance in accelerator physics is the use of machine learning for a wide range of tasks: this is an intriguing and rapidly developing area that promises to have a significant impact on the way that studies in beam dynamics are performed. However, machine learning is a very large subject that would require more space than is available in this book to do it justice. Also, mindful of my own very limited experience and still more limited expertise in this area, it seems advisable to leave it to others to contribute to the texts available on the topic. For similar reasons, I decided also not to include discussion of coherent radiation (whether in the form of coherent synchrotron radiation from dipole magnets, or in free-electron lasers). In any case, there are already a number of excellent textbooks covering this important topic, and there will no doubt be more in the future.

The errors in the first edition are, to my great regret, too numerous to list here. Although they largely consist of minor typographical mistakes, I am all too well aware of the frustration that can result even from just an incorrect minus sign or a missing factor in a term in an equation. I am sincerely grateful to the readers who have taken the trouble to contact me, to let me know when they have found an error. Needless to say, I have taken the opportunity in this edition to correct all those errors of which I am aware, and while it seems overly-optimistic to expect that all the errors have been put right, I hope that there are not too many remaining.

Finally, I must express my thanks to the many readers and colleagues who have contributed to this book, either indirectly through thought-provoking discussions on particular topics, or more directly through comments or questions on sections of the first edition. There are too many people to whom I am indebted in various ways for me to name them all, but I feel that I must thank especially Dr. Bruno Muratori of Daresbury Laboratory, and Prof. Moses Chung of UNIST for many enlightening discussions that have influenced this new edition in a number of ways. My thanks also go, of course, to all who have helped me explore the fascinating field of accelerator beam dynamics.

# Preface to the First Edition

Given a beam of charged particles in an accelerator, the challenge for the accelerator physicist is to explain and control the behaviour of that beam. Beam dynamics provides the tools for describing and understanding the particle motion. The electromagnetic fields that determine the dynamics may arise from components such as multipole magnets or radiofrequency cavities, from stray particles in the vacuum chamber, from the beam itself or from other beams of particles in the accelerator. An understanding of the fields generated by these sources and their effect on the beam dynamics is needed for the design, commissioning and operation of an accelerator.

In calculating the motion of particles in an accelerator, the first step is to select the appropriate physical principles. Here, we appear to be in good shape, since nearly all the important features of beam dynamics can be described and explained using physical laws that have been known for more than a century. In particular, the electromagnetic fields in a beam line must satisfy Maxwell's equations, and the motion of a charged particle through those fields is determined by Hamilton's equations, with an appropriate (relativistic) Hamiltonian. There are certainly some aspects of beam behaviour that give glimpses beyond the regime of purely classical physics: these include the quantum excitation of particle oscillations in a storage ring from synchrotron radiation and effects associated with spin polarisation. But from rather few basic ingredients, there results a rather impressive and somewhat daunting diversity of phenomena. Furthermore, many of the problems that occur in practical situations are rather difficult to solve. Much of the discussion in this book concerns techniques for finding approximate solutions, with certain desirable properties, to the classical equations of motion in various situations in accelerator beam lines. Different techniques are appropriate for different cases.

My main motivation for writing this book (apart from an intrinsic interest in the subject) was to bring together in one place the principles and methods that I have found most important, interesting and useful in my work as an accelerator physicist, dealing with situations that range from the relatively simple to the rather complex. The simpler cases can often be modelled adequately using straightforward techniques based on various approximations. However, most accelerator physicists will sooner or later encounter complicated systems that require accurate descriptions of the beam dynamics: and the simplified techniques cannot always be extended to handle such cases. The approach I have taken in this book is to develop the theory of beam dynamics in such a way that it can be applied, where necessary, to some of the more complex situations encountered in high energy particle accelerators. The aim is to provide a solid foundation for a deeper understanding of the various topics than would be provided by a simple discussion at an introductory level.

Although it is assumed that readers will be familiar with electromagnetism and classical mechanics, these topics are covered briefly in introductory chapters because of the way that nearly all of beam dynamics follows naturally from them. Also, these theories are wide enough that it is worth emphasising the aspects that are of particular relevance to the behaviour of charged particles in accelerators. Without the introductory chapters, this book would be incomplete. Special relativity is not discussed explicitly, but (by definition) is intrinsic to just about all aspects of beam dynamics in high energy accelerators.

In addition to some familiarity with electromagnetism, classical mechanics and special relativity, it is assumed that readers have at least a basic knowledge of the general features and principles of operation of high energy particle accelerators. This includes, for example, the purpose and structure of components such as dipole and quadrupole magnets, solenoids and rf cavities. Although these (and other) components are briefly introduced and described in the appropriate places, my intention is simply to provide a context and to define a starting point for the development of the relevant beam dynamics.

When planning this book, I quickly realised that it would be necessary to make some compromise between the range of topics covered and the depth to which they are treated. The contents are guided by my own experience as an accelerator physicist, and fall into one (or more) of three categories. First, I have included those topics that are widely considered, with some justification, to be fundamental to the study of beam dynamics.



In the second category are results and methods that I use (or see used) on a regular basis. Finally, some subjects are covered that may not be thought essential for practical applications, but are included because they provide some insight into certain aspects of beam behaviour, or simply because they involve interesting physics. My hope is that I have not got the balance between range and depth hopelessly wrong, and that this book will be of some use to those relatively new to the subject, as well as to more experienced accelerator physicists who may find it convenient to have the various topics brought together in one place.

Inevitably, there are important topics that ought to have been included but have been omitted for one reason or another. For example, coherent synchrotron radiation and free-electron lasers are mentioned only in passing; beam cooling (apart from synchrotron radiation damping) is not mentioned at all, nor are ion effects, electron cloud and some other instabilities. The discussions here of spin dynamics, beam-beam effects and numerous other phenomena completely fail to do justice to these interesting and important topics in beam dynamics. Furthermore, regarding those topics that are included, I fear that despite my best efforts at accuracy and consistency, there will inevitably be some errors, for which I apologise in advance.

I faced particular challenges at times in finding appropriate notation. Although I have tried to follow convention as far as possible, there are cases where several different notations or definitions are in use by the accelerator community, or where the conventional notation used in one specific topic conflicts with that used in another. I am afraid that I did not always find satisfactory solutions. I may be fortunate enough to have the opportunity to address some of these deficiencies in the future; but for the time being I just hope that this book, as far as it goes, can be of some use for those with an interest in beam dynamics.

Finally, I should like to acknowledge and thank all those who have helped me in my own efforts to learn accelerator physics. I have had the privilege of working with some wonderful people and outstanding physicists from a number of different laboratories and universities. As I still have a very great deal to learn, I look forward to many future interesting and enlightening discussions.