KoPAS 2015

Lecture 4

Basics of RF Cavities

Moses Chung (UNIST)
mchung@unist.ac.kr

Lecture 4 Introduction

- In general, charged particles are focused and bent by use of magnets, and accelerated by use of electromagnetic waves in cavities.

$$\mathbf{F} = q\mathbf{E} = q\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right)$$

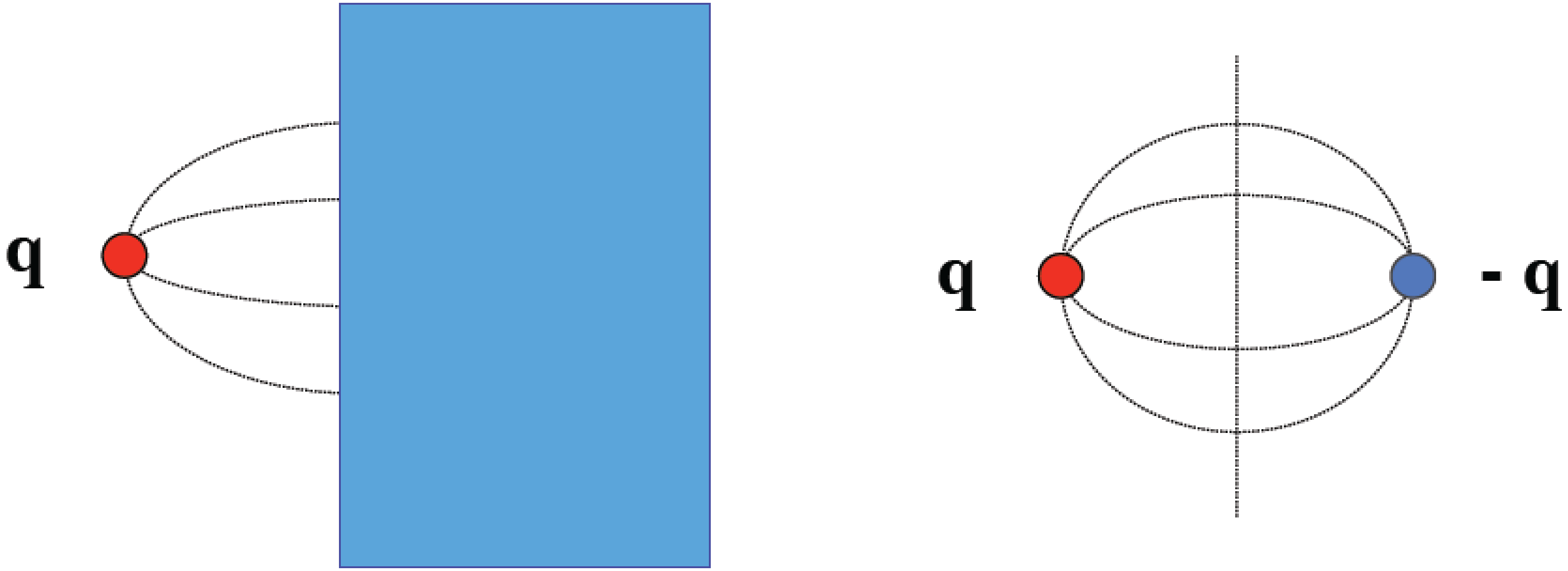
- DC acceleration is limited by high-voltage sparking and breakdown. It is very difficult to produce DC voltages more than a few million volts.

- RF accelerators bypass this limitation by applying a harmonic time-varying electric field to the beam, which is localized into bunches, such that the bunches always arrive when the field has the correct polarity (phase) for acceleration.

- The beam is accelerated within electromagnetic-cavity structures, in which a particular electromagnetic mode is excited from a high-frequency external power source.

Boundary Conditions (Revisited)

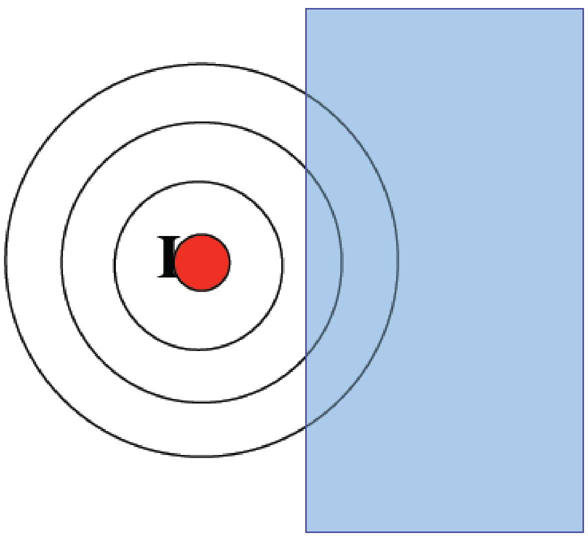
- Electric fields near a good conductor: For both static and time-varying cases, the electric filed lines are perpendicular to the surface.



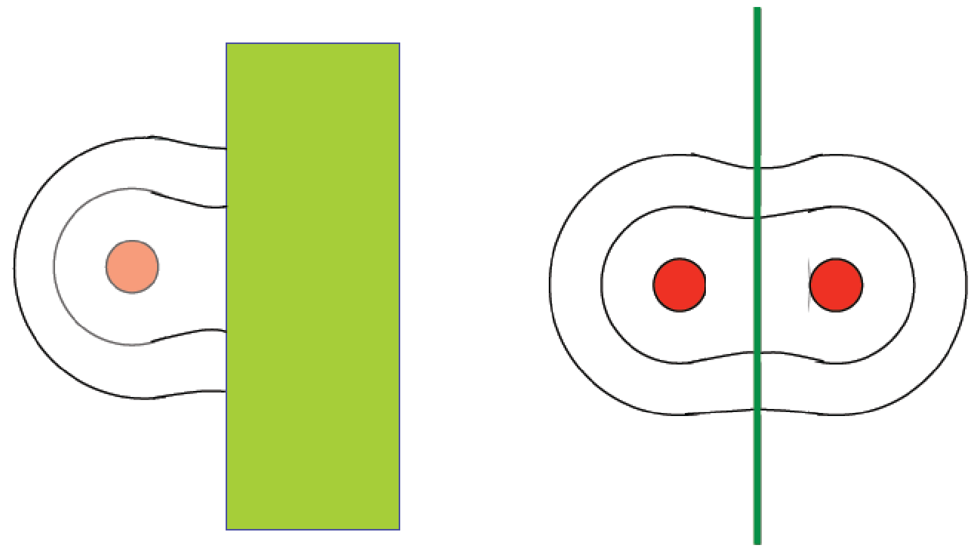
Boundary Conditions (Revisited)

- Static magnetic fields:

$\mu_r \approx 1$



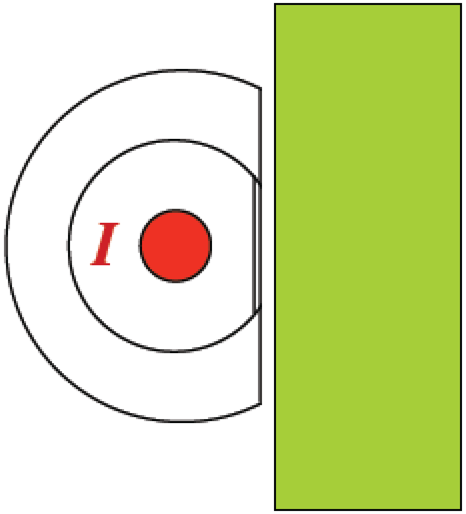
$\mu_r \gg 1 \quad \text{(ferromagnetic)}$

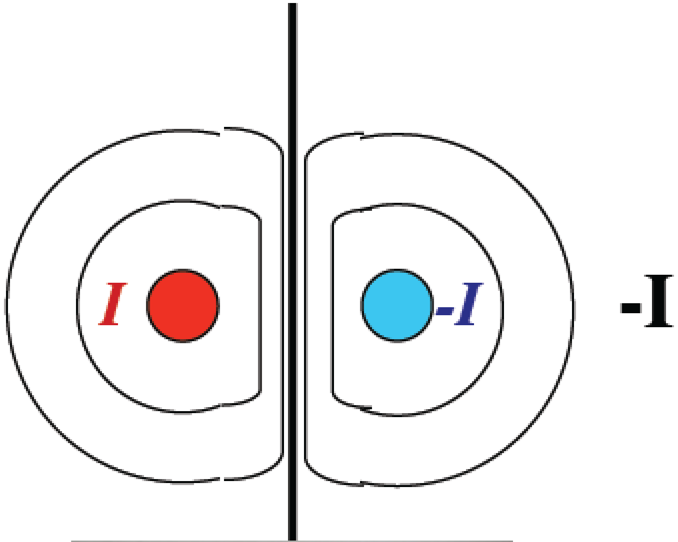


Boundary Conditions (Revisited)

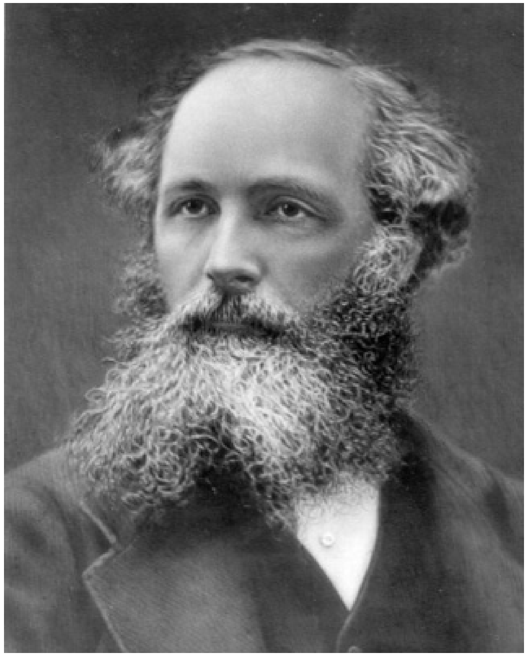
- Time-varying magnetic fields:

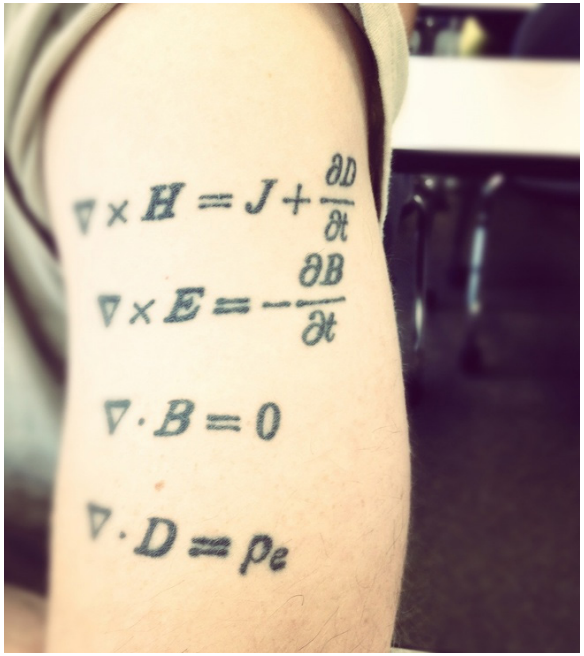
Please use right-hand rules
for both currents !





Electromagnetism of Waveguides/Cavities





Helmholtz Equations

- We assume all fields and sources have a time dependence $e^{-i\omega t}$ (or $e^{j\omega t}$)

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r})e^{-i\omega t}\right], \quad \mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{B}(\mathbf{r})e^{-i\omega t}\right]$$

- $\mathbf{E}(\boldsymbol{r})$ and $\mathbf{B}(\boldsymbol{r})$ are generally complex (i.e., may have phase shift), and time derivative is

$$\frac{\partial}{\partial t} \longrightarrow -i\omega$$

- For source free cases,

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = i\omega \mathbf{B}, \quad \nabla \times \mathbf{B} = -i\omega\mu_0\epsilon_0\mathbf{E}$$

- Helmholtz equations with wave number $k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c}$:

$$\nabla^2\mathbf{E} + k_0^2\mathbf{E} = 0, \quad \nabla^2\mathbf{B} + k_0^2\mathbf{B} = 0$$

- In terms of vector potential with Lorentz gauge:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \quad \longrightarrow \quad \nabla^2\mathbf{A} + k_0^2\mathbf{A} = 0$$

Plane Wave in Free Space

- For free space (no boundary):

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right], \quad \mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

- Phase velocity:

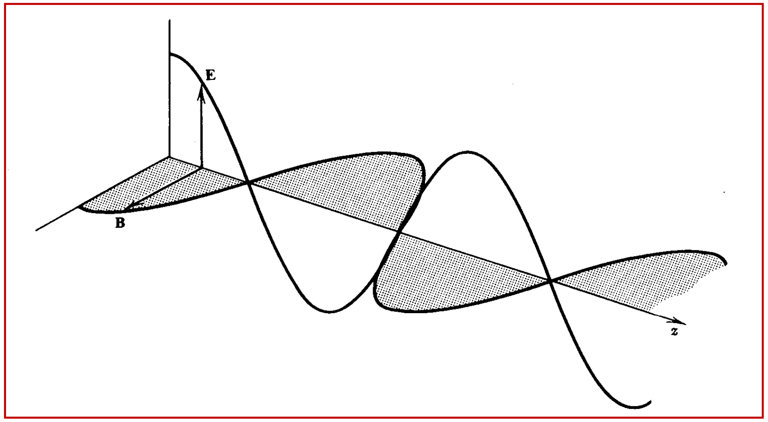
$$v_{ph} = \frac{\omega}{k_0} = \frac{\omega}{|\mathbf{k}|} = \frac{\omega}{\sqrt{k_x^2+k_y^2+k_z^2}} = c$$

- Consequences of Maxwell equations:

$$\nabla\cdot\mathbf{E}=0 \longrightarrow \mathbf{k}\cdot\mathbf{E}=0, \quad \nabla\cdot\mathbf{B}=0 \longrightarrow \mathbf{k}\cdot\mathbf{B}=0$$

$$\nabla\times\mathbf{E}=i\omega\mathbf{B} \longrightarrow \mathbf{k}\times\mathbf{E}=\omega\mathbf{B}, \quad \nabla\times\mathbf{B}=-i\omega\mu_0\epsilon_0\mathbf{E} \longrightarrow \mathbf{k}\times\mathbf{B}=-\frac{\omega}{c^2}\mathbf{E}$$

$$\boxed{\mathbf{E}\perp\mathbf{k}, \quad \mathbf{B}\perp\mathbf{k}, \quad \mathbf{E}\perp\mathbf{B}, \quad E=cB}$$



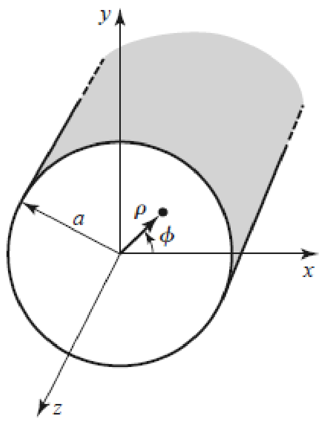
TM Mode Solution

- From conducting boundary, electromagnetic wave can be transformed into TM (Magnetic field is Transverse to z) mode.
- TM fields can be found from one vector component of the magnetic vector potential (note that $\nabla \cdot \mathbf{A} \neq 0$):

$$\mathbf{A} = A_z \hat{z}$$

- In cylindrical coordinates:

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right) A_z + k_0^2 A_z = 0$$



- Separation of variables with arbitrary constant C (complex in general):

$$A_z = C \times J_m(k_\rho \rho) \cos(m\phi) e^{\pm i k_g z}$$

$$-k_g^2 + k_0^2 = k_\rho^2$$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \left(k_\rho^2 - \frac{m^2}{\rho^2}\right)\right] J_m(k_\rho \rho) = 0$$

TM Mode Solution

- Field components can be expressed by A_z alone:

$$\mathbf{B} = \nabla \times \mathbf{A} \longrightarrow B_\rho = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}, \quad B_\phi = -\frac{\partial A_z}{\partial \rho}, \quad B_z = 0$$

$$\mathbf{E} = \frac{i}{\omega \mu_0 \epsilon_0}\nabla \times \mathbf{B} \longrightarrow E_\rho = -\frac{i}{\omega \mu_0 \epsilon_0}\frac{\partial B_\phi}{\partial z}, \quad E_\phi = +\frac{i}{\omega \mu_0 \epsilon_0}\frac{\partial B_\rho}{\partial z}$$

$$E_z = \frac{i}{\omega \mu_0 \epsilon_0}[\nabla \times (\nabla \times \mathbf{A})]_z \longrightarrow E_z = \frac{i}{\omega \mu_0 \epsilon_0}\left[\frac{\partial^2 A_z}{\partial z^2} - \nabla^2 A_z\right] = \frac{i}{\omega \mu_0 \epsilon_0} k_\rho^2 A_z$$

- Boundary conditions:

$$E_\phi(\rho = a) = E_z(\rho = a) = B_\rho(\rho = a) = 0$$

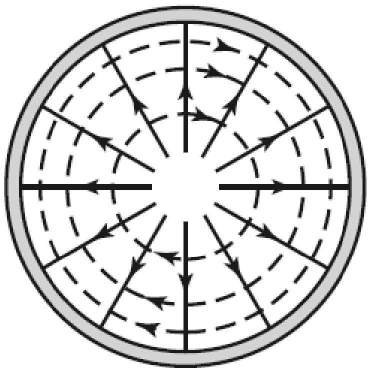
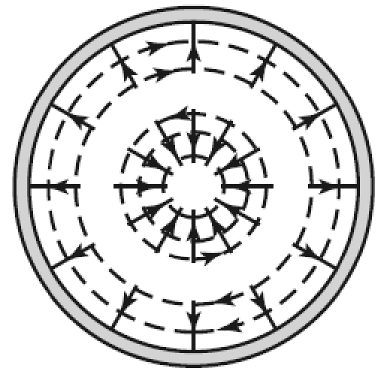
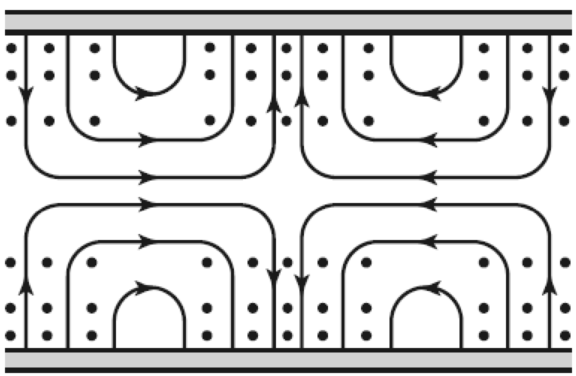
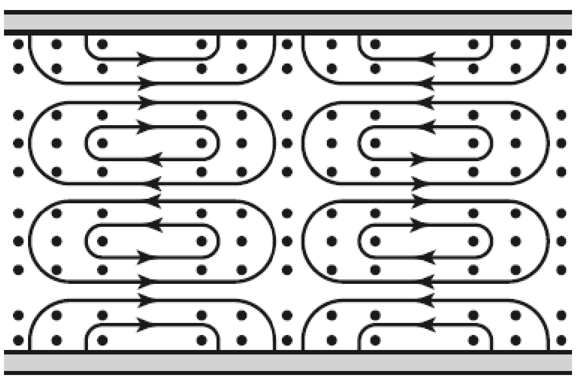
$$J_m(k_\rho a) = 0 \longrightarrow k_\rho = \frac{x_{mn}}{a} = \frac{\omega_c}{c}$$

  - x_{mn} : n-th zero of the Bessel function of order m. (e.g., $x_{01} = 2.405$)

- Dispersion relation for guide propagation constant and wavelength:

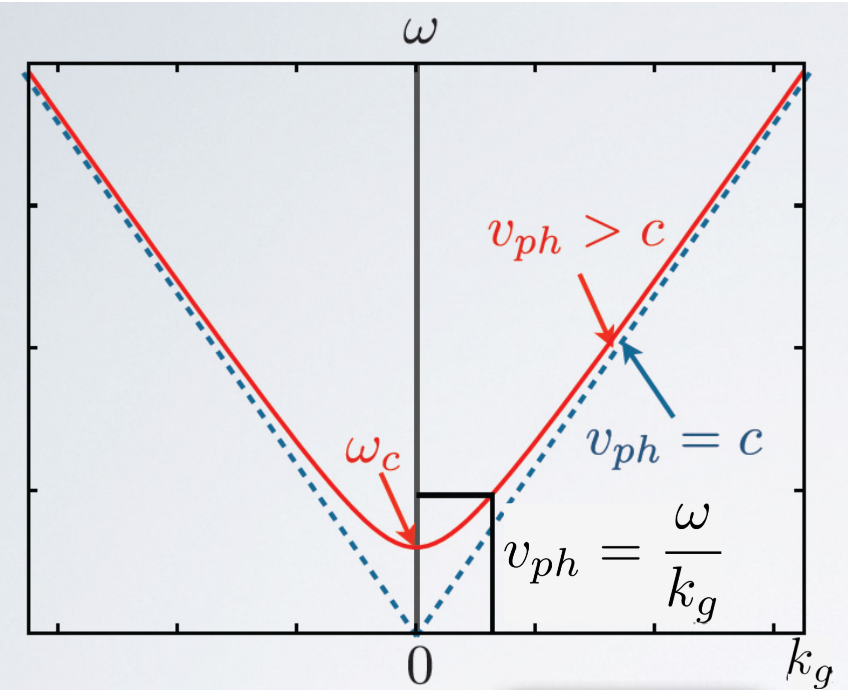
$$k_g^2 = k_0^2 - k_\rho^2 \longrightarrow \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda_g}\right)^2 + \frac{\omega_c^2}{c^2}$$

Example

TM_{01}	TM_{02}
	
	

Dispersion Relation

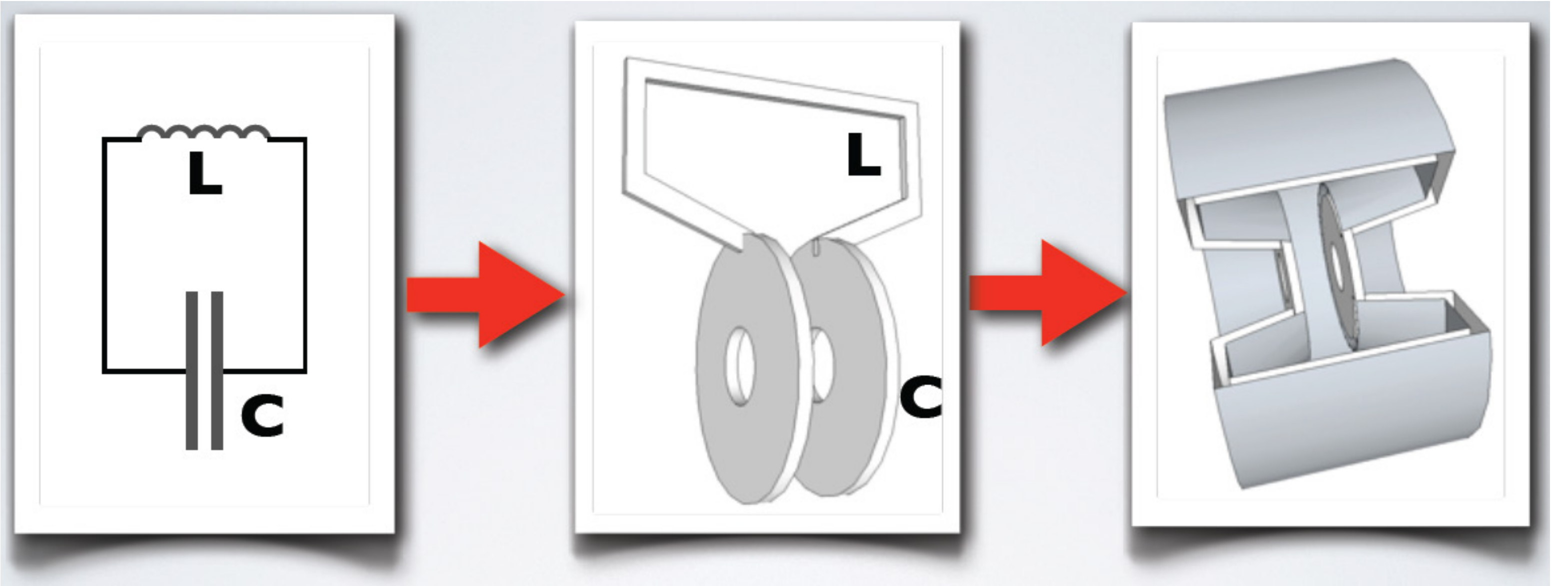
$$\omega^2 = c^2 k_g^2 + \omega_c^2$$



- There is a "cut-off frequency", below which a wave will not propagate. It depends on dimensions.
- At each excitation frequency is associated a phase velocity ω/k_g, , the velocity at which a certain phase travels in the waveguide.
- Energy (and information) travel at group velocity $d\omega/dk_g$, which is between 0 and c. This velocity has respect the relativity principle!
- Synchronism with RF (necessary for acceleration) is impossible because a particle would have to travel at $v = v_{ph} > c$!
- To use the waveguide to accelerate particles, we need a "trick" to slow down the wave. → Cavities (with reflections)

Cavity

- The capacitance from the fact that change on the wall of the cavity will induce charge on the center conductor just like a capacitor.
- The inductance comes from the fact that as current is induced to flow along the walls of the cavity, this produces an axial magnetic field in the same way as an inductor.
- The resistance comes from the resistance in the copper walls.

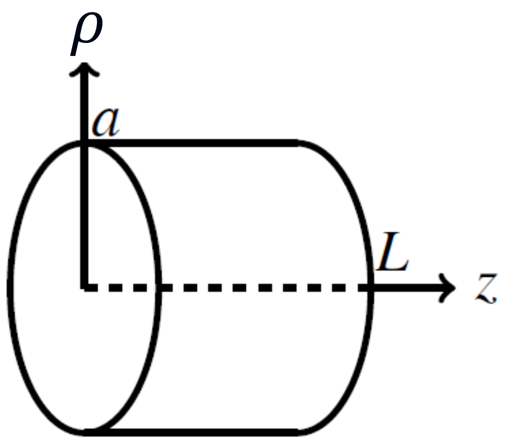


$$\omega_{res} = 2\pi f_{res} = \frac{1}{\sqrt{LC}}$$

TM Mode of Pillbox Cavity

- We simply superpose two waves in a circular waveguide, one propagating in the positive z direction and the other propagating in the negative z.

$$A_z = C \times J_m(k_\rho \rho)\cos(m\phi)\left(e^{+ik_g z} + e^{-ik_g z}\right) = 2C \times J_m(k_\rho \rho)\cos(m\phi)\cos(k_g z)$$



- Additional boundary conditions at $z = 0$ and $z = L$:

$$E_\rho(z=0) = E_\phi(z=0) = E_\rho(z=L) = E_\phi(z=L) = 0$$

$$E_\rho, E_\phi \propto \sin(k_g z) \longrightarrow k_g L = p\pi \;\; (p = 0, 1, 2, \cdots)$$

- Dispersion relation: discrete resonance frequency

$$\frac{\omega^2}{c^2} = \left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{p\pi}{L}\right)^2$$

Example: TM_{010} Mode

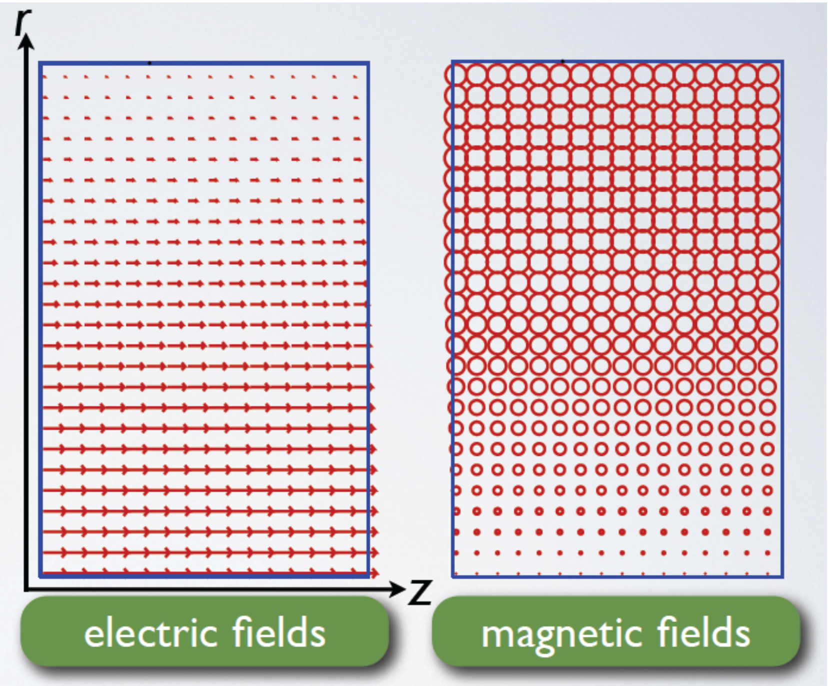
- Simplest and lowest frequency mode: $TM_{mnp} = TM_{010}$

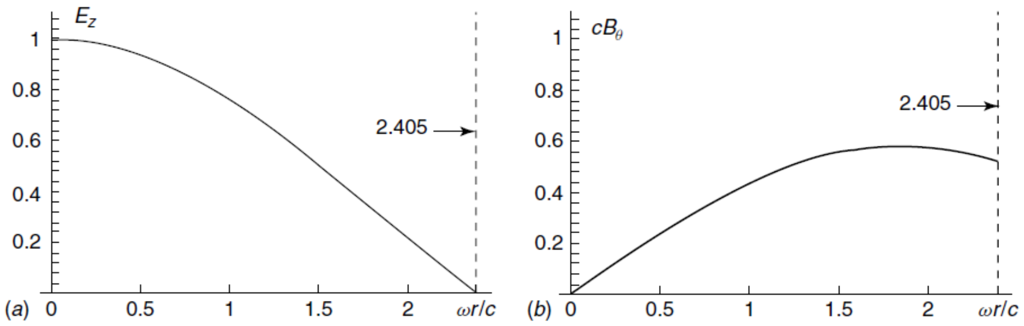
$$k_\rho = \frac{2.405}{a}, \quad \omega = \omega_{010} = \frac{2.405c}{a}$$

- Explicit expression for fields:

$$E_z = E_0 J_0(k_\rho \rho) e^{-i\omega t}, \quad B_\phi = -i\frac{E_0}{c} J_1(k_\rho \rho) e^{-i\omega t}$$

Phase difference

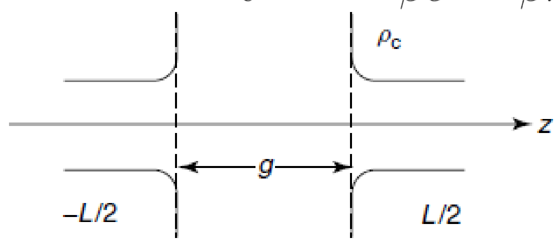


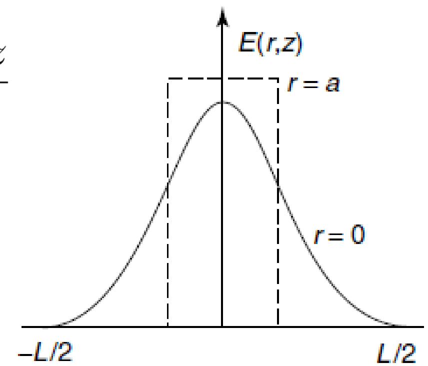


Cavity Parameters: Transit Time Factor

- We suppose that the field is symmetric about $z = 0$, and confined within an axial distance L containing the gap, in which velocity change is small.

$$\omega t \approx \omega \frac{z}{v} = 2\pi f \frac{z}{\beta c} = \frac{2\pi z}{\beta \lambda}$$





$$\Delta W = q \int_{-L/2}^{L/2} E(0,z)\cos(\omega t + \varphi)dz = qV_0 T \cos\varphi$$

where

$$V_0 = \int_{-L/2}^{L/2} E(0,z)dz = E_0 L, \quad T = \frac{\int_{-L/2}^{L/2} E(0,z)\cos(\omega t)dz}{V_0} \approx \frac{\int_{-L/2}^{L/2} E(0,z)\cos(2\pi z/\beta\lambda)dz}{V_0}$$

- Accelerating voltage and gradient: Effect of transit time factor (T) is included.

$$V_{acc} = V_0 T, \quad E_0 T \ [\mathrm{MV/m}] = \frac{V_{acc}}{L}$$

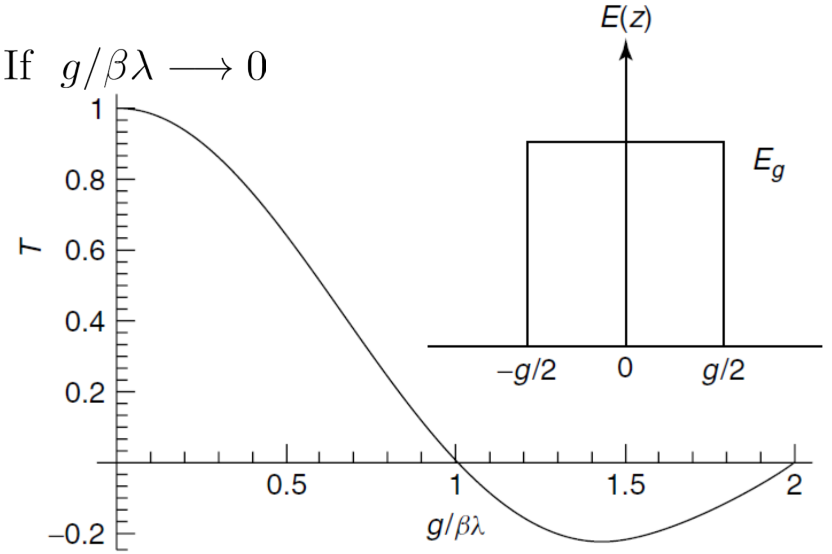
Cavity Parameters: Transit Time Factor

- Physical meaning: ratio of the energy gained in the time-varying RF field to that in a DC field of voltage $V_0 \cos(\varphi)$.
- Thus, T is a measure of the reduction in the energy gain caused by the sinusoidal time variation of the field in the gap.

Ex] A simple TM_{010} pillbox cavity of length g:

$$E(0,z) = E_g = const., \quad T = \frac{\sin(\pi g/\beta\lambda)}{\pi g/\beta\lambda}$$

If $g/\beta\lambda \longrightarrow 0$



Cavity Parameters: Shunt Impedance

- A figure of merit that measures the effectives of producing an axial voltage V_0 for a given power dissipated P.

$$R_s = \frac{V_0^2}{P}$$

- Including the transit time factor, we define effective shunt impedance:

$$R = \frac{(V_0 T)^2}{P}$$

- Be careful ! Accelerator community uses different definition of the shunt impedance.

$$R_s^{circuit} = \frac{V_0^2}{2P}$$

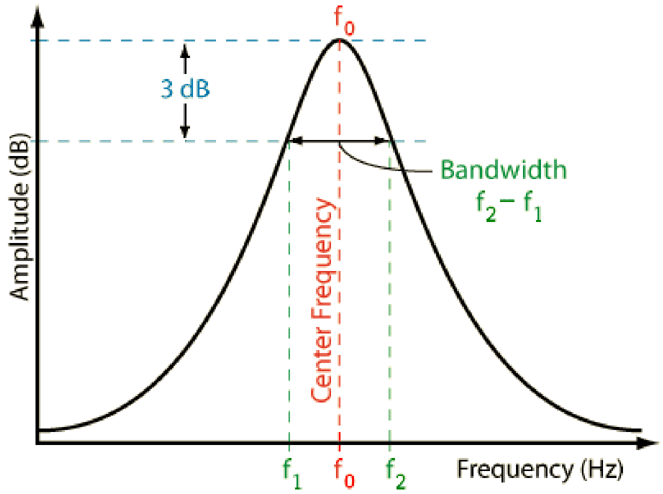
- R-over-Q: the ratio of R to Q (quality factor), which measures the efficiency of acceleration per unit stored energy U at a given frequency.

$$\left[\frac{R}{Q}\right] = \frac{(V_0 T)^2}{\omega U}$$

  - A single geometric quantity given in Ohms.

Cavity Parameters: Quality Factor

- The quality factor Q describes the bandwidth of a resonator and is defined as the ratio of the reactive power (stored energy) to the real power that is lost in the cavity walls.



$$Q = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f} = \frac{\omega_0 U}{P}$$

Ex] For SC cavities, $Q \approx 10^{10} \sim 10^{11}$. Why so high ?

- Filling/Decay time of a cavity:

$$P = -\frac{dU}{dt} = \frac{\omega_0 U}{P} \quad \longrightarrow U(t) = U_0 e^{-2t/\tau}, \quad \tau = \frac{2Q}{\omega_0}$$

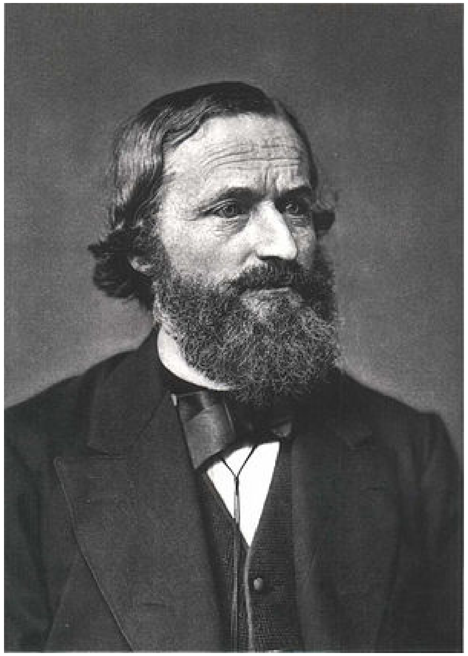
- If the cavity is connected with a power coupler, some power will leak out though the coupler and be dissipated through the external load/waveguide.

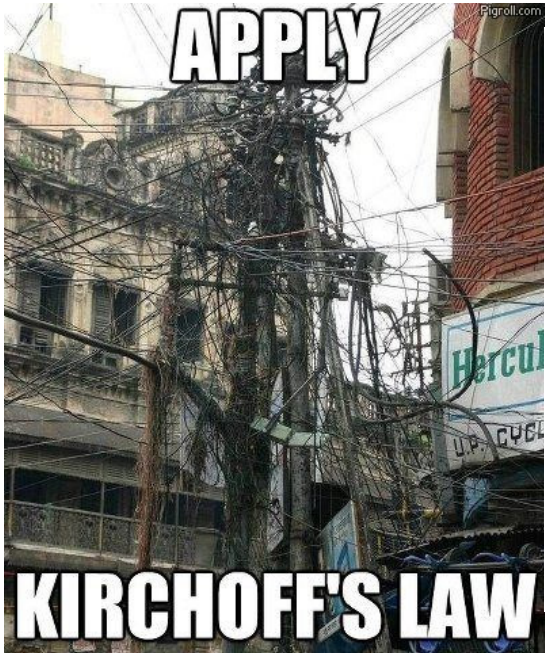
$$Q_{ext} = \frac{\omega_0 U}{P_{ext}}, \quad Q_{loaded} = \frac{\omega_0 U}{P_{ext} + P_{cav}}$$

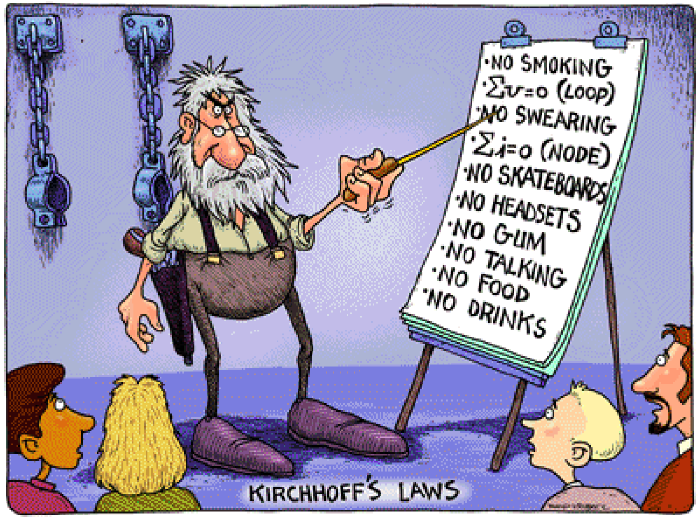
[Homework]

- For a pillbox cavity of length L and radius a operated in the fundamental mode (i.e., TM_{010} mode), express the following parameters in terms of geometric parameters, surface resistance R_{surf} of the cavity wall, and/or basic physical constants.
  - Resonance frequency
  - Quality factor
  - Effective shunt impedance
  - R-over-Q

Circuit Theory of Cavities

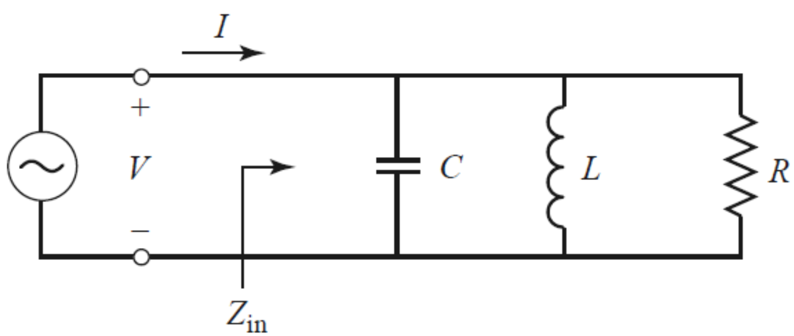






Resonant Circuit

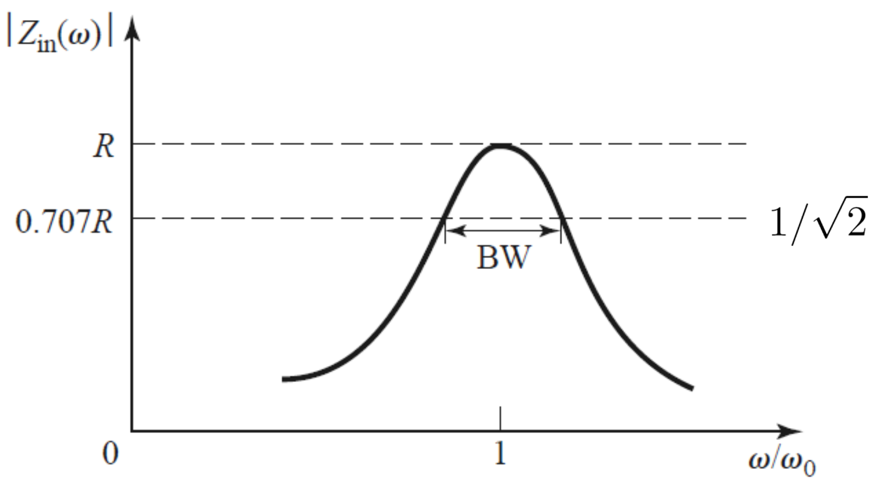
- A parallel resonant circuit driven by a current generator is the simplest model for describing a single mode of an accelerating cavity (damped driven oscillator).



(a)

$$I(t) = C\frac{dV}{dt} + \frac{1}{L}\int V\,dt + \frac{V}{R}$$

$$V(t) = \underbrace{\left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)^{-1}}_{=Z_{\text{in}}} I(t)$$



(b)

- Resonance frequency:

$$\omega_0 = 1/\sqrt{LC}$$

- Stored energy at resonance ($U_m = U_e$):

$$U = CV_0^2/2 = L|I_L|^2/2$$

- Dissipated power:

$$P = V_0^2/2R$$

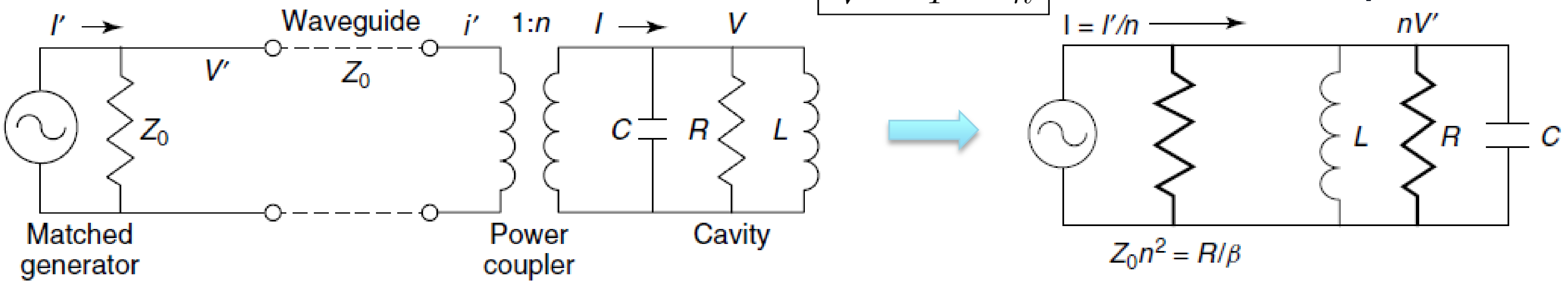
- Quality factor:

$$Q = \omega_0 U/P = \omega_0 RC = R/\omega L$$

Equivalent Circuit for Generator/Cavity

- Transformer coupling:

$$\frac{V'}{V} = \frac{I}{I'} = \frac{1}{n}$$



$$P_{ext} = \frac{V'^2}{2Z_0} = \frac{V^2}{2Z_0 n^2} \quad \longrightarrow \quad Z_0 \rightarrow Z_0 n^2$$

- Coupling coefficient: a measure of the waveguide-to-cavity coupling strength.

$$\beta_c = \frac{P_{ext}}{P_{cav}} = \frac{Q_{cav}}{Q_{ext}} = \frac{\omega_0 RC}{\omega_0 n^2 Z_0 C} = \frac{R}{Z_0 n^2}$$

- Loaded Q:

$$\frac{1}{Q_{loaded}} = \frac{1}{Q_{cav}} + \frac{1}{Q_{ext}} = \frac{1}{Q_{cav}}(1 + \beta_c)$$

  – Note that loaded Q become ½ of the unloaded Q for critically coupled case ($\beta_c = 1$).

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Reflection Coefficient

- A steady-state reflection coefficient produced by the cavity load impedance:

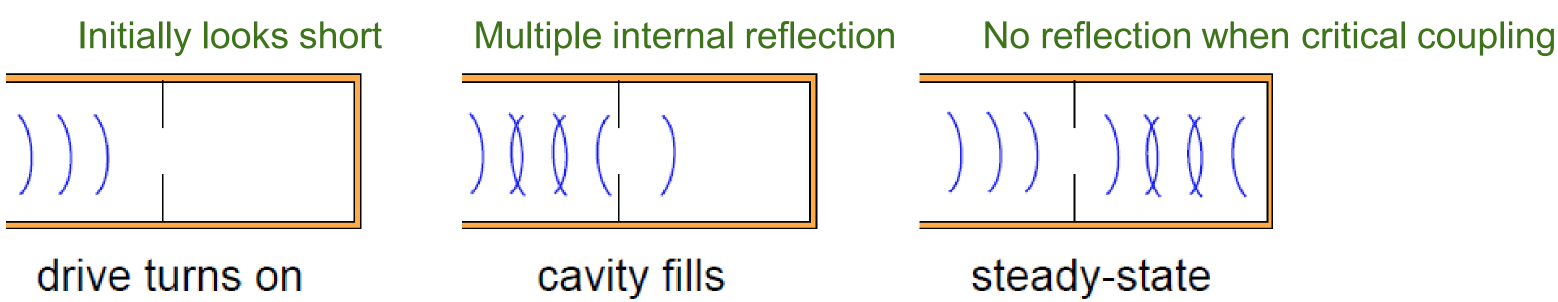
$$\Gamma = \frac{\text{Reverse Voltage}}{\text{Forward Voltage}} = \frac{V_-}{V_+} = \frac{Z_L - Z_0}{Z_L + Z_0} \xrightarrow{\text{at resonance}} \frac{\beta_c - 1}{\beta_c + 1}$$

- A time-dependent reflection coefficient:

$$\Gamma(t) = \frac{\text{Reverse Voltage}}{\text{Forward Voltage}} = \frac{V_-(t)}{V_+}$$

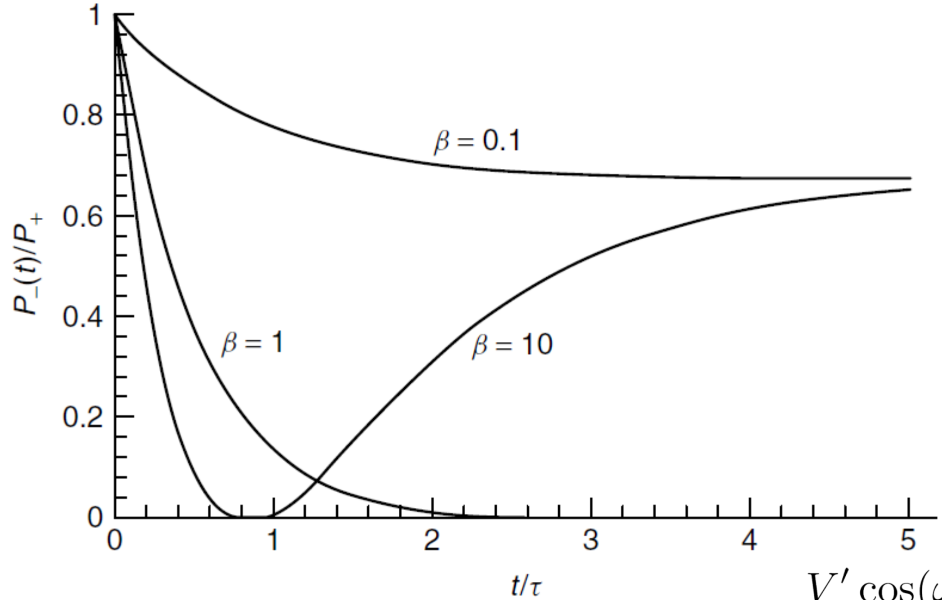
  - Voltage in the waveguide: $V' = V_+ + V_-$
  - Voltage in the cavity: $V = nV' = n(V_+ + V_-)$

- The reverse voltage in the input waveguide is indeed the sum of the two (out of phase) travelling waves; direct-reflected wave from the coupler + radiated (diffracted) wave for the cavity.

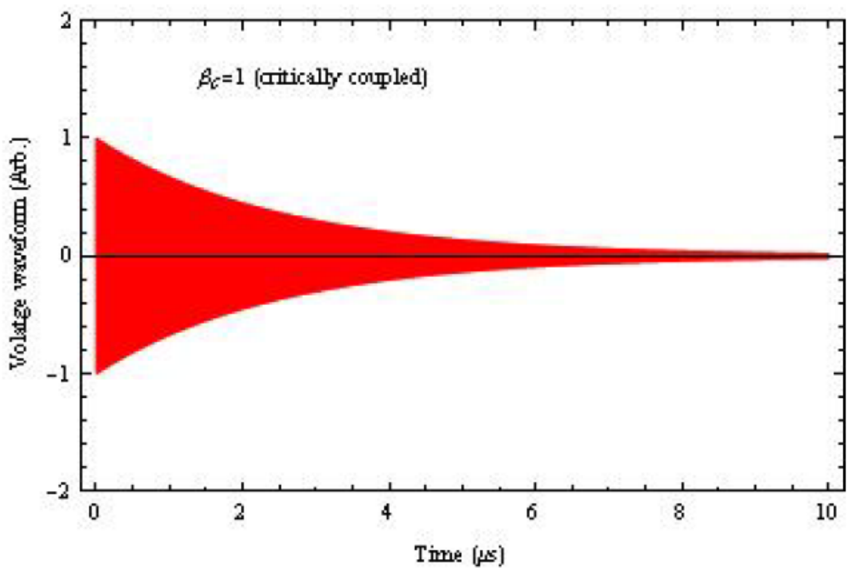


Example: Turn on at $t = 0$

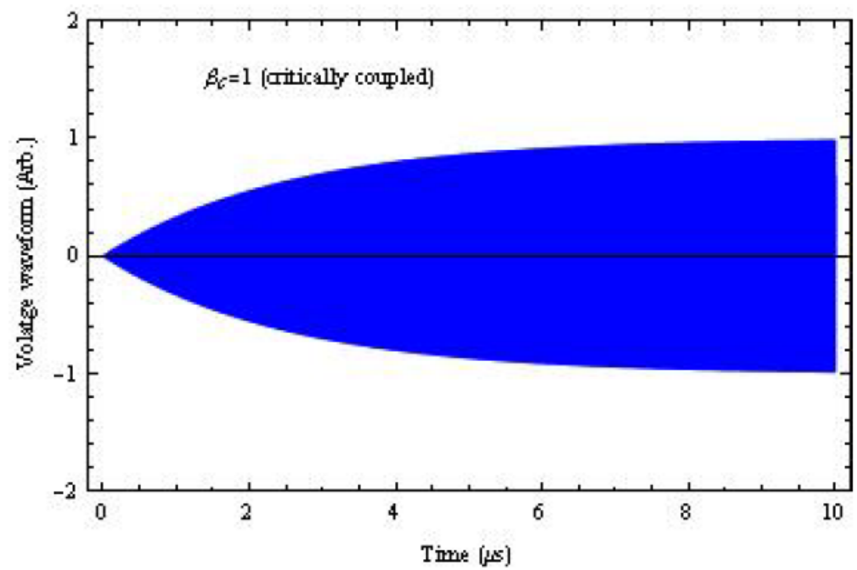
$\Gamma^2(t) =$



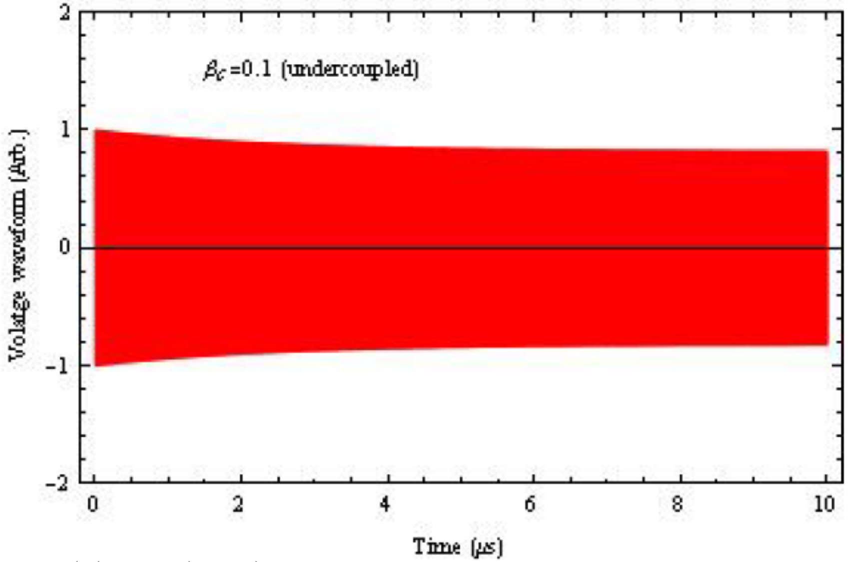
$V_-(t)\cos(\omega_0 t)$

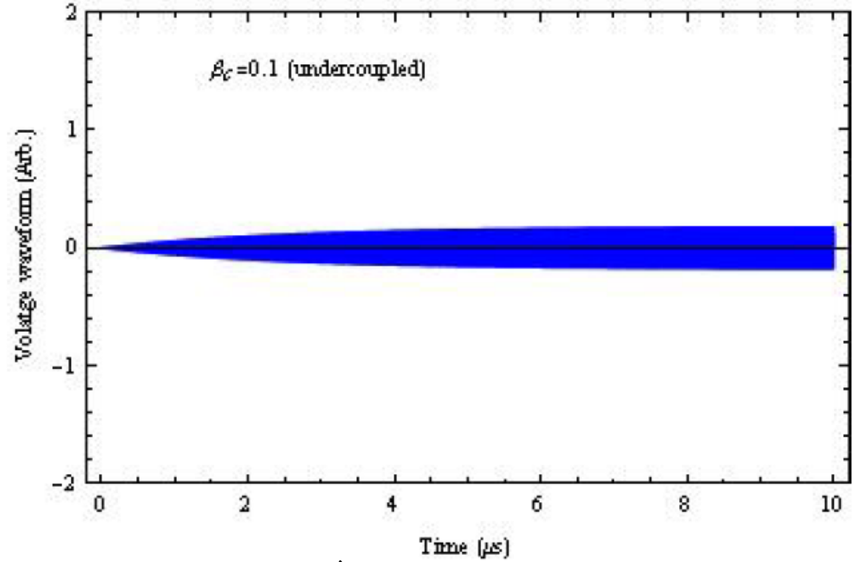


$V' \cos(\omega_0 t) = (V_-(t) + V_+)\cos(\omega_0 t)$



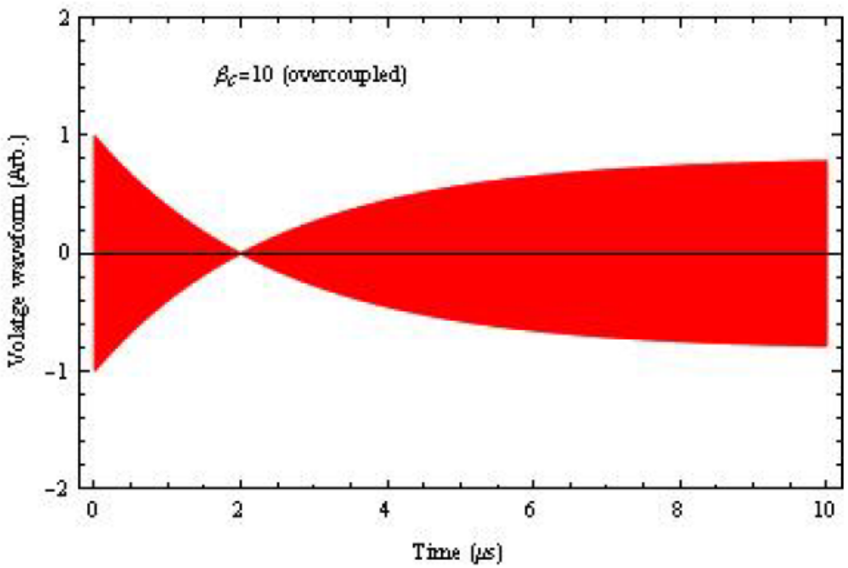
Example: Turn on at $t = 0$

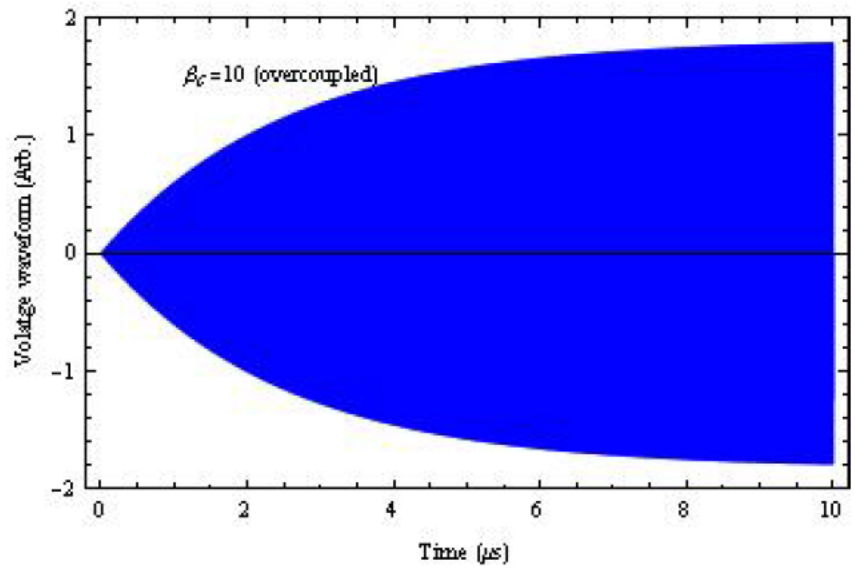




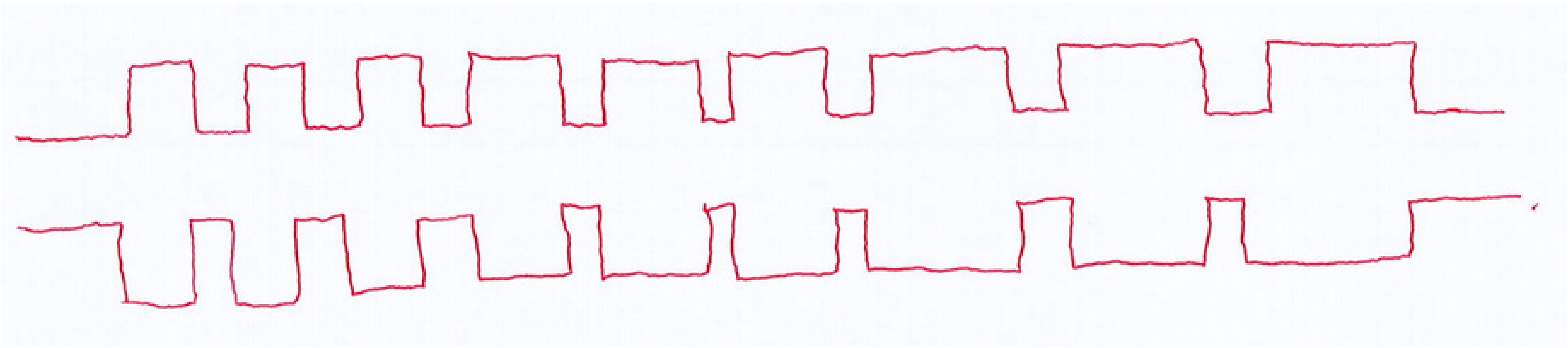
$$V_-(t)\cos(\omega_0 t)$$

$$V'\cos(\omega_0 t) = (V_-(t) + V_+)\cos(\omega_0 t)$$





More Realistic Accelerating Cavities



Who drew this ?

Drift Tube vs Waveguide

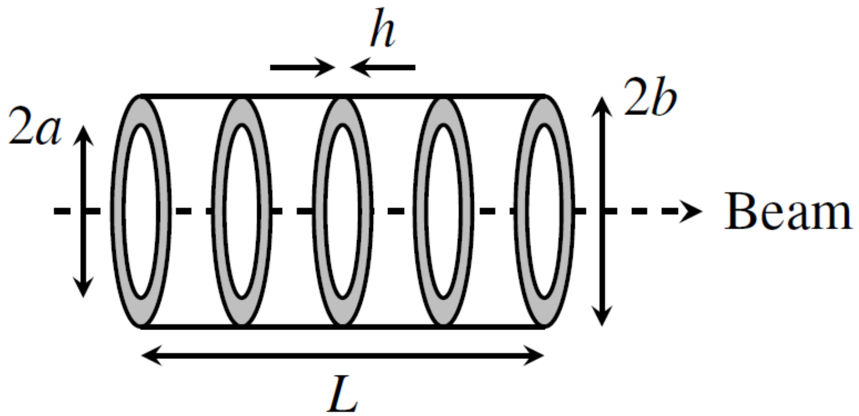
- The Alvarez linac (see lecture 3) operates at the standing wave TM_{010} mode, with drift tubes used to shield the electric field at the decelerating phase.
- The effective acceleration gradient is reduced by the transit time factor and the time the particle spends inside the drift tube.

- A periodically-loaded waveguide accelerator, where the phase velocity is equal to the particle velocity, can effectively accelerate particles in its entire length.
- A waveguide accelerator is usually more effective if the particle velocity is high, and there are two ways to operate high-β cavities: standing wave or traveling wave.

- The standing wave normally operates at $\pi/2$ mode, where $d\omega/dk$ has its highest value (no mode-hopping to nearby resonances. See Brillouin diagram).
- However, the shunt impedance in $\pi/2$ mode operation is reduced by a factor of 2, because only half of the cavity cells are used for particle acceleration.

Disk-Loaded Waveguide

- In order to slow down the waves in simple waveguide, we introduce some periodic obstacles. Iris acts as a scatter, resulting in a transmitted as well as a reflected wave.



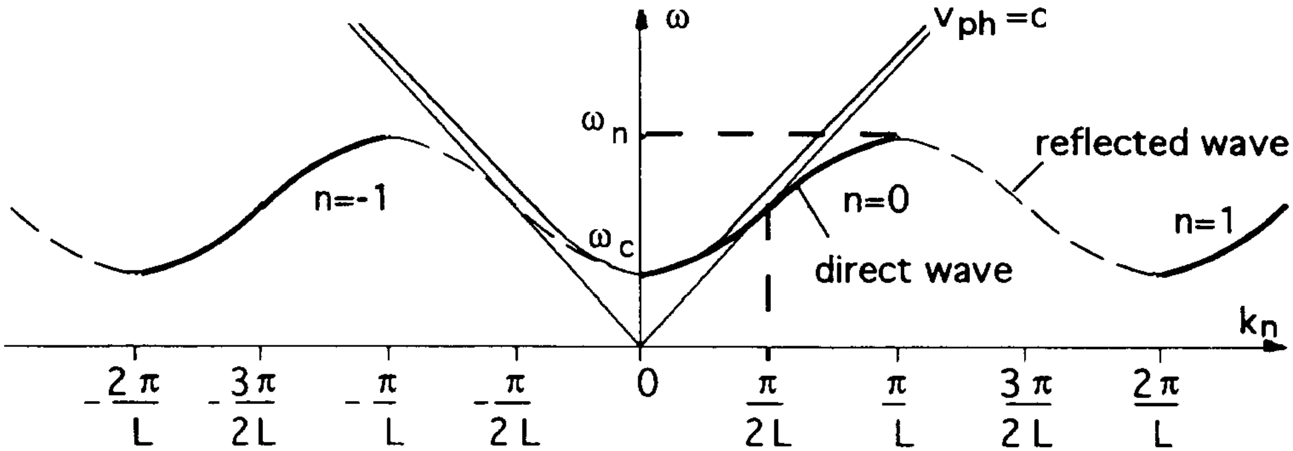
- The complicated boundary conditions cannot be satisfied by a single mode, but by a whole spectrum of space harmonics.

- From Chap.3.11 of Wangler's textbook:

$$\omega = \frac{2.405c}{b}\sqrt{1+\kappa[1-\cos(k_n L)e^{-\alpha h}]}$$

$$\kappa = \frac{4a^2}{3\pi J_1^2(2.405)b^2 L} \ll 1, \quad \alpha \approx \frac{2.405}{a}$$

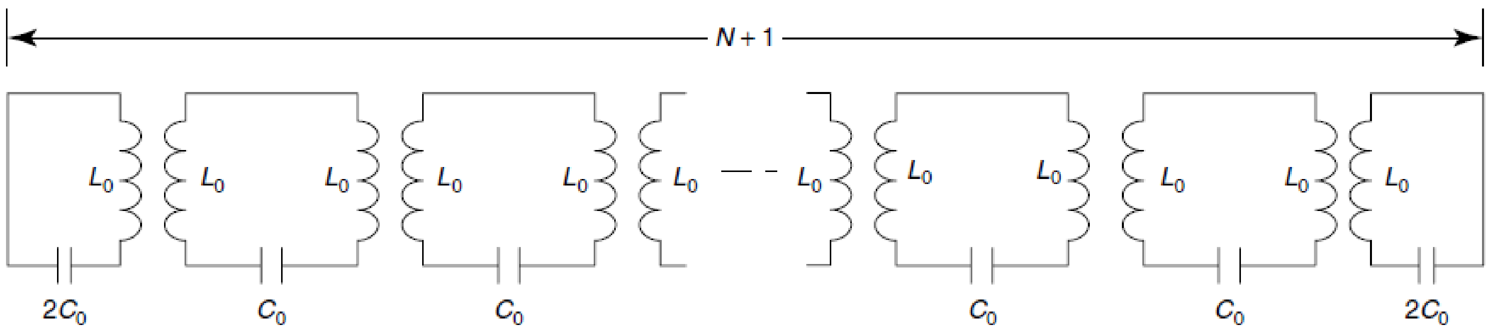
Brillouin Diagram

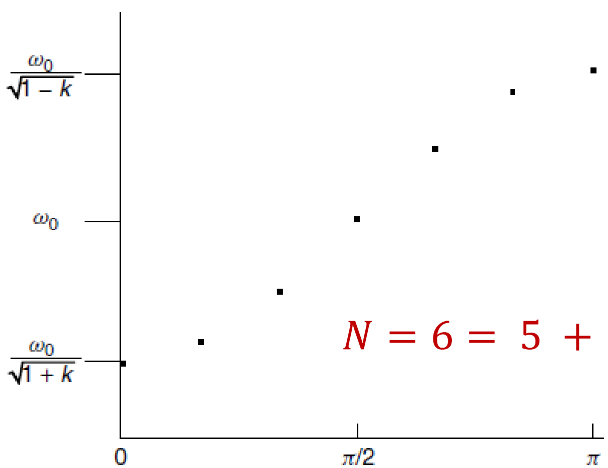


- For a given mode, there is a limited passband of possible frequencies; at both ends of the passband, the group velocity is 0.
- For a given frequency, there is an infinite series of space harmonics ($-\infty < n < +\infty$). All space harmonics have the same group velocity, but different v_{ph}.
- The directed (reflected) wave are characterized by $v_g > 0$ ($v_g < 0$), i.e., the EM energy flows in the $+z$ ($-z$) direction.
- At the end of the waveguide, the EM energy can either be dissipated into a matched load (travelling-wave structure) or be reflected back and forth by shortening end walls (standing-wave structure).

SW Structure as Coupled Oscillators

- One can obtain a cylindrical standing wave structure by simply closing both ends of a disk-loaded circular waveguide with electric walls.
- Owing to the additional boundary conditions in the longitudinal direction, we obtain another restriction (i.e., discrete frequencies and discrete phase changes) on the existence of electromagnetic modes in the structure.





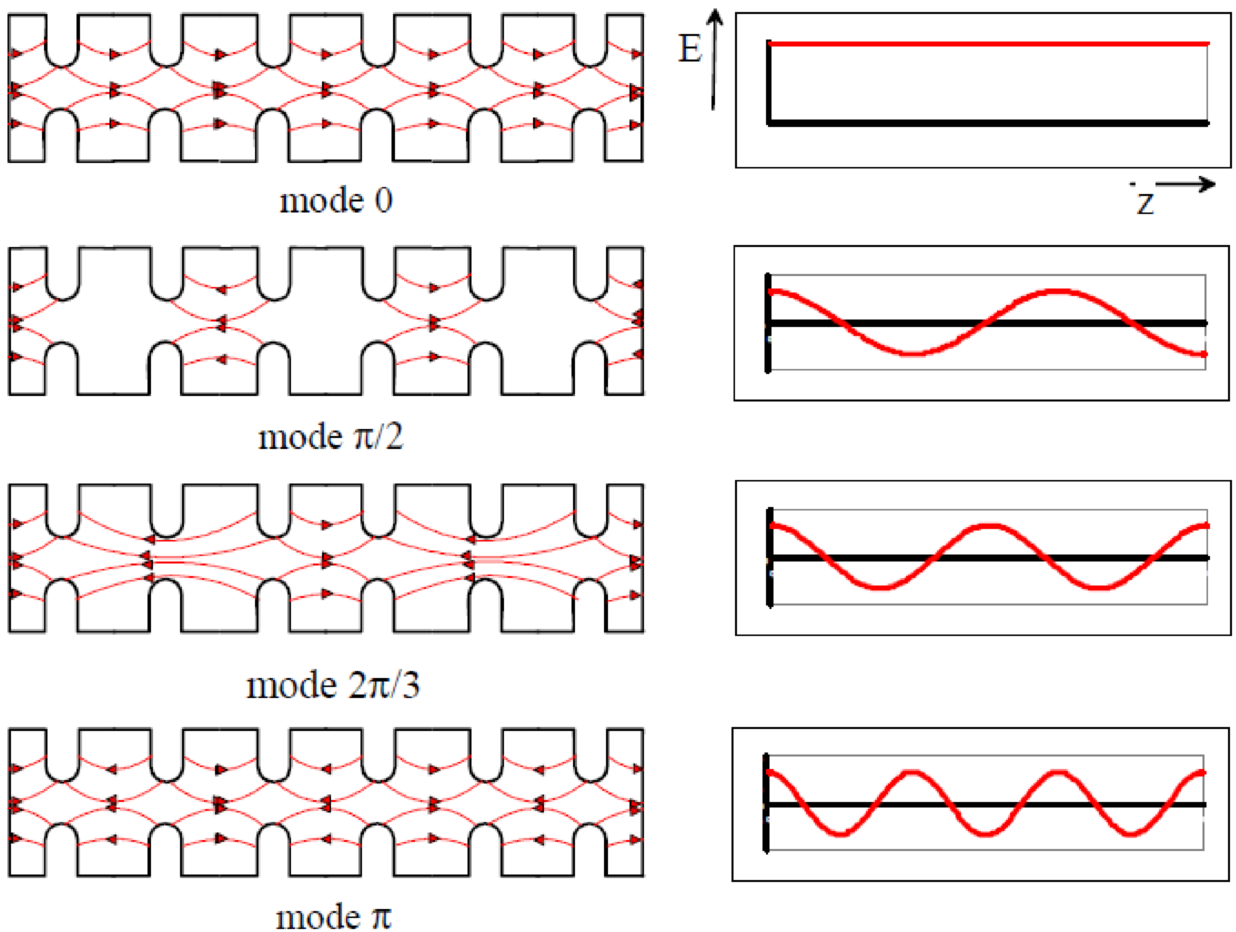
$$N = 6 = 5 + 2 \times \frac{1}{2}$$

$$\omega_n = \frac{\omega_0}{\sqrt{1 + \kappa \cos(n\pi/N)}}, \quad n = 0, 1, \cdots, N$$

- From Chap.4.10 of Wangler's textbook.
- ω_0 is the frequency of the $\pi/2$ mode and of an uncoupled single cell.
- κ is the cell-to-cell coupling constant.

Standing Wave Structure

- For acceleration, the particles must be in phase with the E-field on axis.
- Boundary at both ends is that E-field must be perpendicular to the plane.



$l = \beta\lambda$

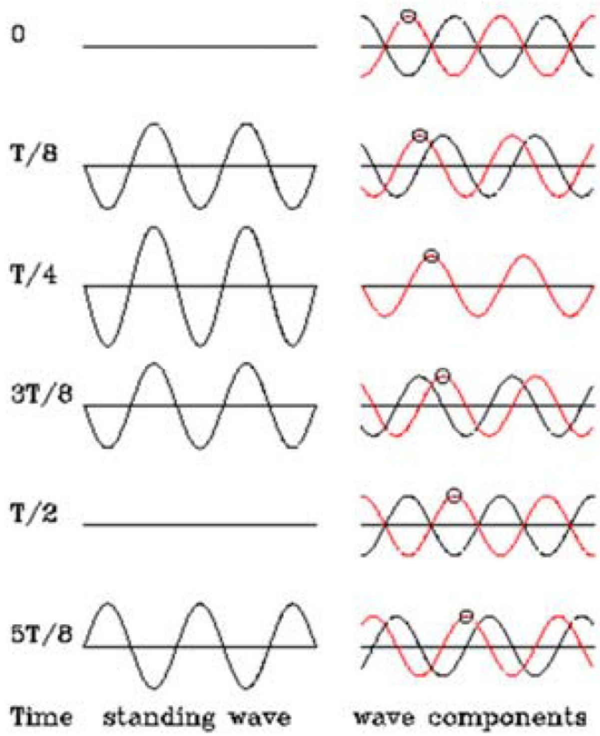
$l = \beta\lambda/4$

$l = \beta\lambda/3$

$l = \beta\lambda/2$

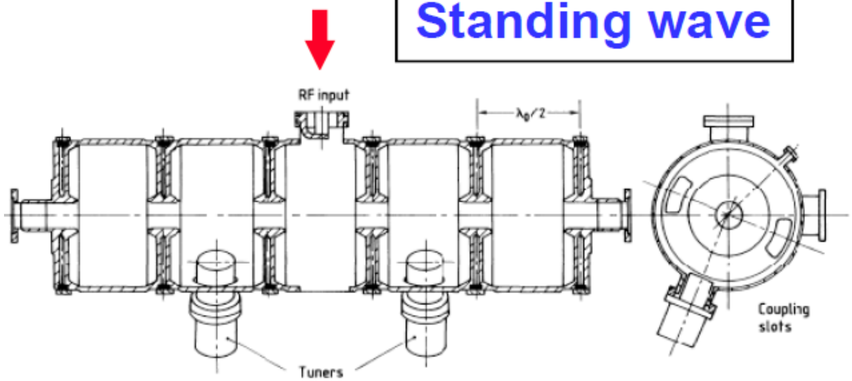
Standing Wave Structure

- These standing wave modes are generated by the sum of 2 traveling waves in opposite directions.
- Since only the forward wave can accelerate the beam, the shunt impedance (effectiveness of producing axial voltage for a given power dissipated) is ½ of that of the travelling wave structure.
- The standing wave could accelerate oppositely charged beams traveling in opposite directions.



SW vs TW Structure

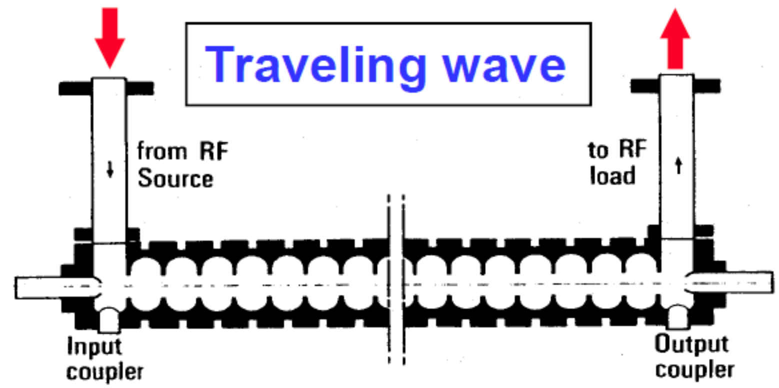
Standing wave



Chain of coupled cells in SW mode.
Coupling (bw. cells) by slots (or open). On-axis aperture reduced, higher E-field on axis and power efficiency.
RF power from a coupling port, dissipated in the structure (ohmic loss on walls).
Long pulses. Gradients 2-5 MeV/m

Used for Ions and electrons at all energies

Traveling wave



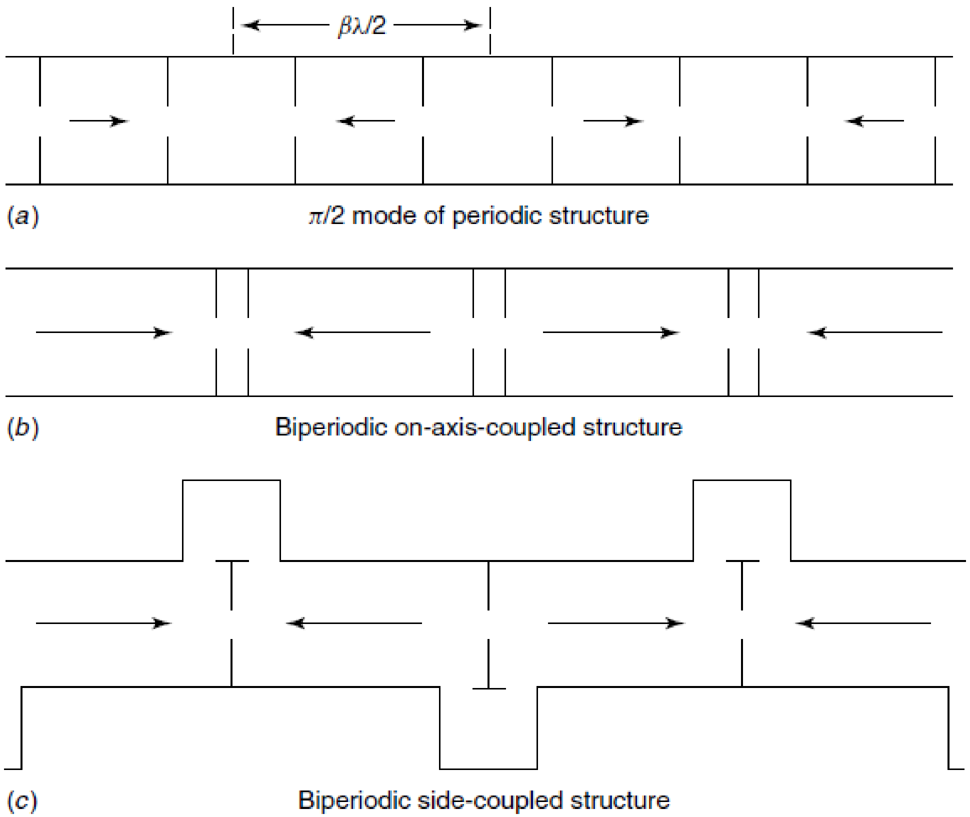
Chain of coupled cells in TW mode
Coupling bw. cells from on-axis aperture.
RF power from input coupler at one end, dissipated in the structure and on a load.
Short pulses, High frequency (≥ 3 GHz).
Gradients 10-20 MeV/m
Long structures, with no space for transverse focusing
Used for Electrons at v~c

Comparable RF efficiencies

[From Maurizio Vretenar (CERN)]

Coupled-Cavity Linacs

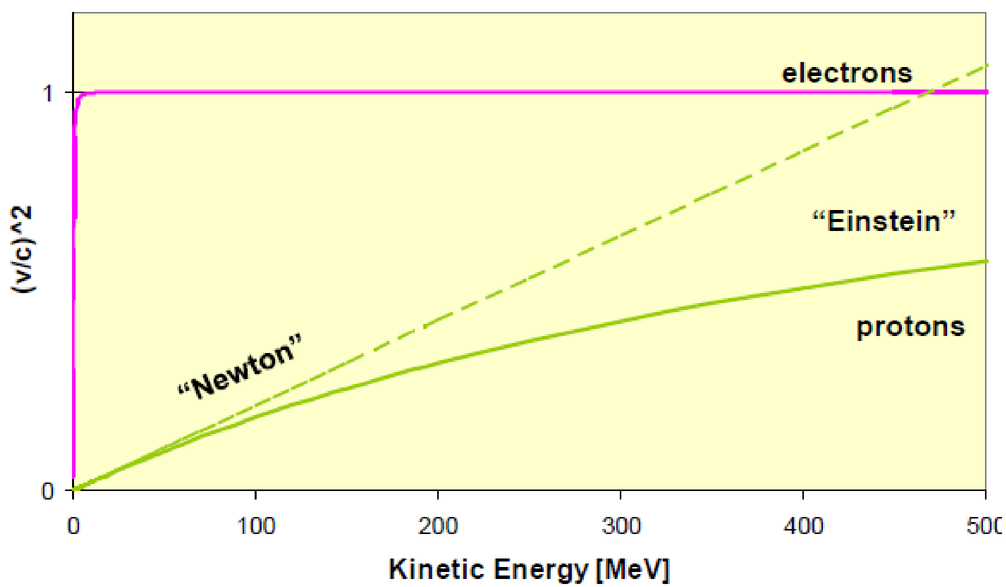
- Since every other cavity cell has no electric fields in $\pi/2$ standing wave operation, these empty cells can be shortened or moved outside.
- Such a design regains the other half of the shunt impedance and provides very efficient proton beam acceleration for $\beta > 0.3$.



Proton vs Electron Acceleration

- Protons change their velocity up to the GeV range ($\beta = 0.95$ at $W = (\gamma - 1)mc^2 = 2$ GeV):
  - Accelerating structures (distance between gaps) need to be adapted to the changing velocity.

- Electrons are almost immediately relativistic ($\beta = 0.95$ at $W = (\gamma - 1)mc^2 = 1.1$ MeV):
  - Basically from the source onwards one can use the same accelerating structure (optimized for β=1.0) for the rest of the linac.



Nearly the same velocity, but increasing energy

Lecture 4 References

- Basic Electromagnetism:
  - R. K. Wangsness, *Electromagnetic Fields* (Wiley, 1986)
  - J. D. Jackson, *Classical Electrodynamics* (Wiley, 1998)
- Beam and Accelerator Physics:
  - J. B. Rosenzweig, *Fundamentals of Beam Physics* (Oxford, 2003)
  - M. Conte and W. M. MacKay, *An Introduction to the Physics of Particle Accelerators* (World Scientific, 2008)
  - T. P. Wangler, RF Linear Accelerators (Wiley, 2008)
- RF Technology:
  - R. Pasquinelli and D. McGinnis, *Microwave Measurements and Beam Instrumentation Lab.* (USPAS, 2012)
  - F. Gerigk, *RF Basics I and II* (CAS, 2013)
  - D. M. Pozar, *Microwave Engineering* (Wiley, 2012)

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