Lecture 1

Basics of Electromagnetism, Classical Mechanics, and Relativity

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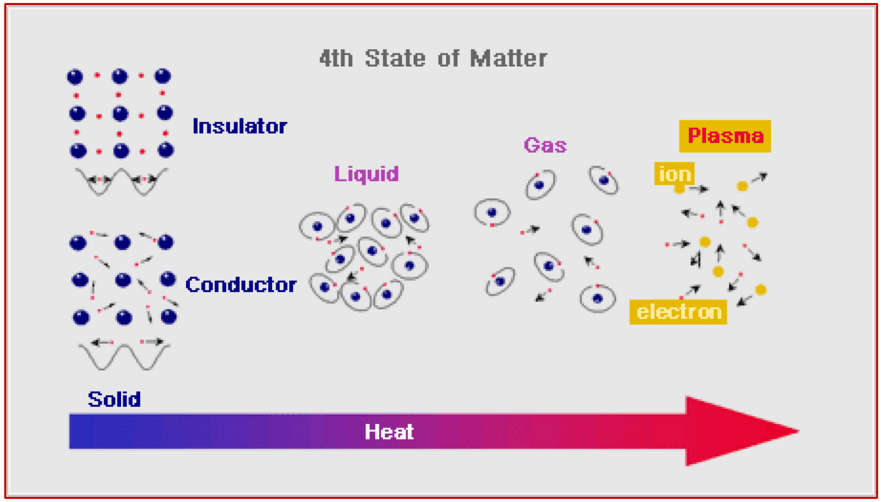
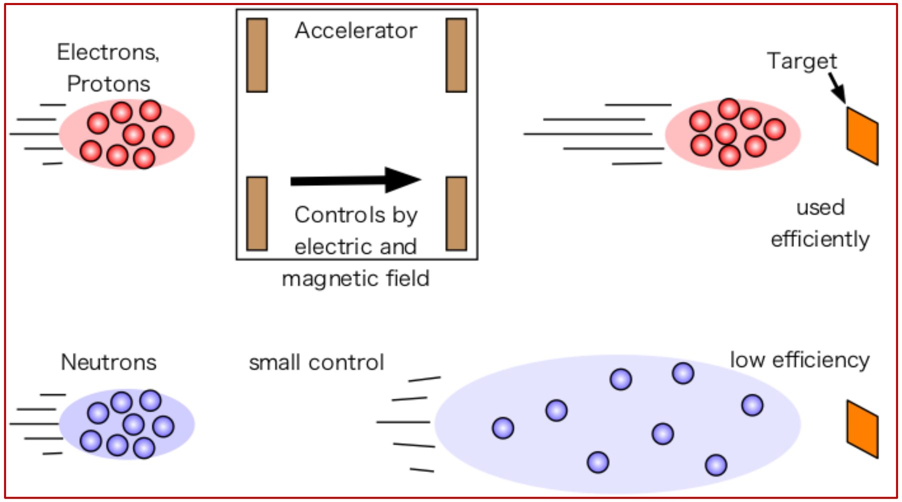
Lecture 1 Introduction

- Accelerator physics is a branch of applied physics that deals with all the physics issues associated with accelerators.
- The goal is the production of energetic particle beams for other applications (~beam physics).
- Particle beams are collections of charged particles all travelling in nearly the same direction with nearly the same speed (possibly relativistic).

- Accelerator physics encompasses broad disciplines, ranging from engineering and technology to diagnostics/controls, to experimental physics, to computer science, to computational and theoretical physics.
- Accelerator physics assumes basic knowledge in electromagnetism, classical mechanics, and special theory of relativity. Also basic understanding on magnet/RF/microwave engineering would be helpful.

- In general, charged particles are focused and bent by use of magnets, and accelerated by use of electromagnetic waves in cavities.
- Accelerator physics studies motions of charged particles under the influence of electromagnetic fields within the context of classical physics.



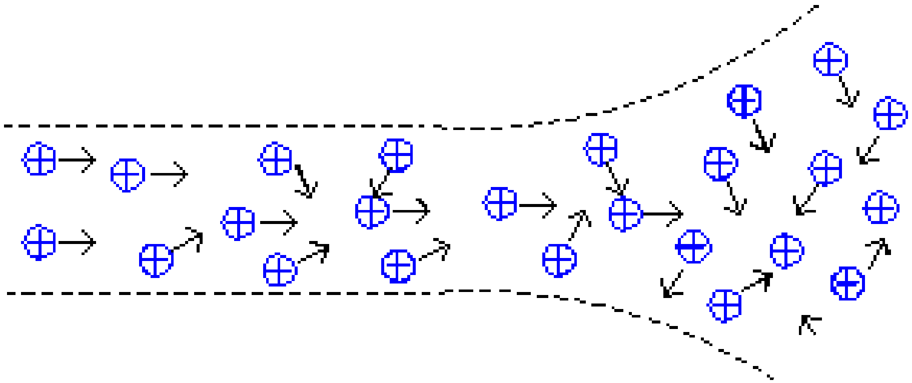


To a large degree, accelerator physics and plasma physics are quite similar. Both involve nonlinear dynamics (single-particle effects) and collective instabilities (multi-particle effects). However, there is an important difference:

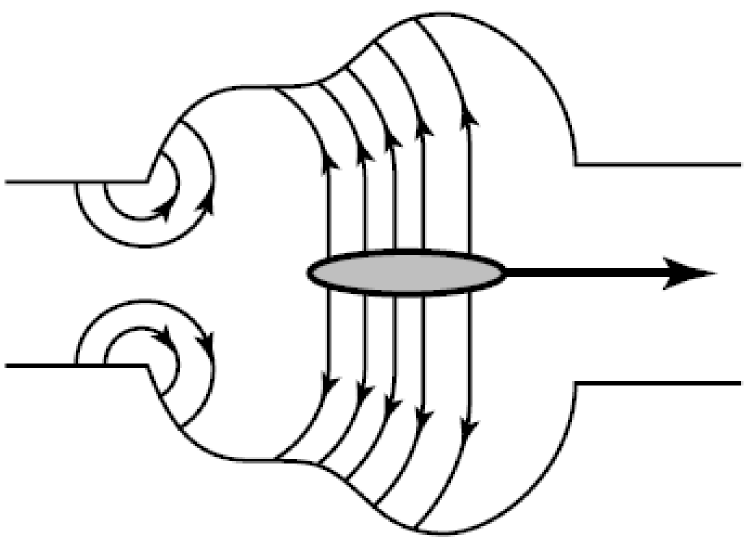
$$\text{beam self fields} > \text{external applied fields} \qquad \text{(plasma)}$$
$$\text{beam self fields} \ll \text{external applied fields} \qquad \text{(accelerators)}$$

[From A. Chao (SLAC)]

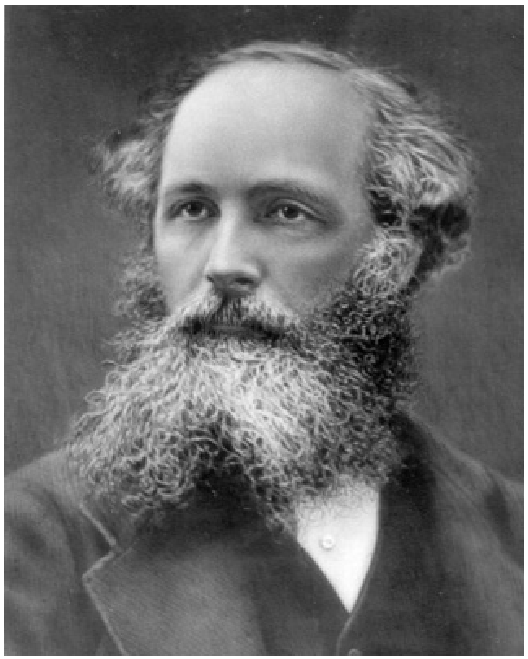
Space charge effects → Nonlinearity, resonance, coupling

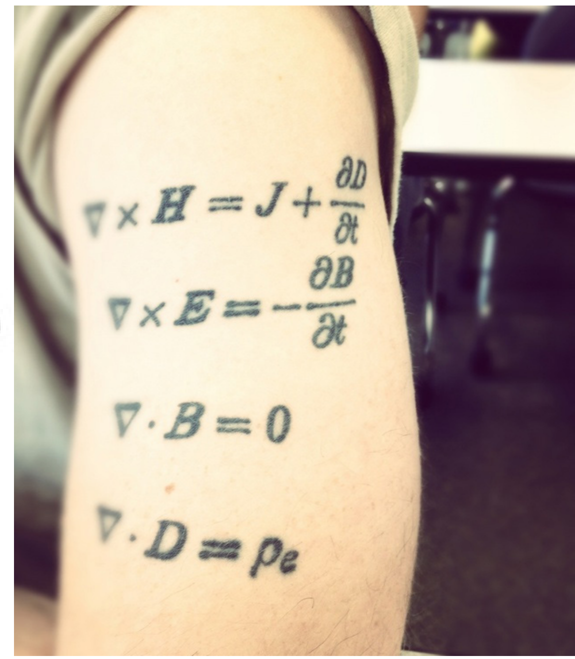


Pipe with Structure
→ Wakefields



Electromagnetism





Maxwell Equations

- Classical electrodynamics is governed by the Maxwell equations. In the SI (MKS) system of units, the equations are

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}$$

- For external sources in vacuum, the constitutive equations are

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu_0$$

- The equations are linear: the sum of two solutions, $\mathbf{E}_1, \mathbf{B}_1$ and $\mathbf{E}_2, \mathbf{B}_2$ is also a solution corresponding to the sum of densities $\rho_1 + \rho_2,\ \mathbf{J}_1 + \mathbf{J}_2$.

Charge and Current Densities

- The free electric charge density and current density are related by the equation of continuity, which is implicit in the Maxwell equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

- For a point charge moving along a trajectory $\boldsymbol{r} = \boldsymbol{r}_0(t)$,

$$\rho = q\delta[\mathbf{r} - \mathbf{r}_0(t)], \quad \mathbf{J} = q\mathbf{v}\delta[\mathbf{r} - \mathbf{r}_0(t)]$$

$$\mathbf{v} = \frac{d\mathbf{r}_0(t)}{dt}$$

- Note that

$$\delta[\mathbf{r} - \mathbf{r}_0(t)] = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

Scalar and Vector Potentials

- It is often convenient to express the fields in terms of the vector and the scalar potentials (two homogeneous Maxwell equations are automatically satisfied).

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- The potentials are not uniquely specified.

$$\phi \to \phi - \frac{\partial \Lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A} + \nabla\Lambda$$

- We can choose a set of potentials to satisfy the so-called Lorentz condition.

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

- The Lorentz condition results in the symmetric and decoupled form of the inhomogeneous wave equations.

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\rho/\epsilon_0, \quad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}$$

Coulomb Gauge

- Within a closed region of space containing no free charges, surrounded by an equipotential surface (e.g., RF cavities):

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \to \phi = const. = 0$$

- The Lorentz condition becomes Coulomb gauge:

$$\nabla \cdot \mathbf{A} = 0$$

- The electric and magnetic fields are obtained from the vector potential alone.

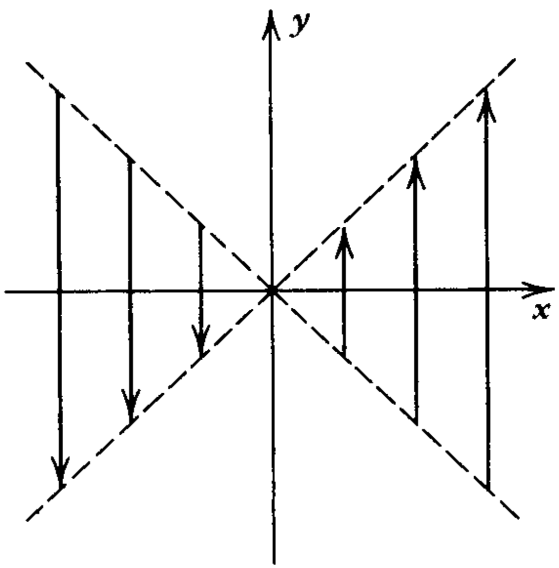
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- For time-independent case (e.g., Magnets),

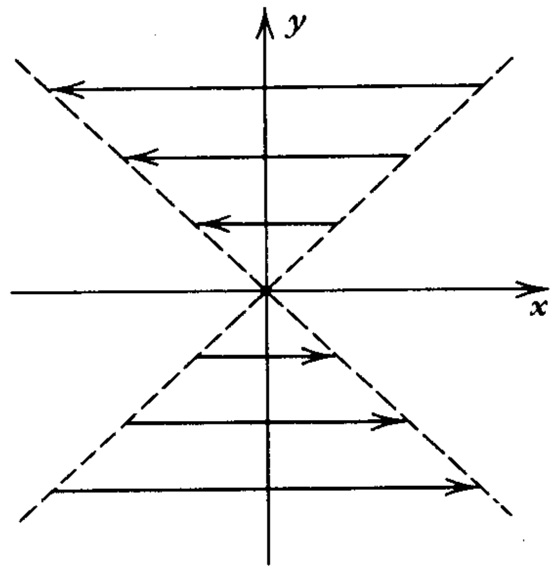
$$\mathbf{E} = 0, \quad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \to \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv$$

Example: Uniform Magnetic Induction

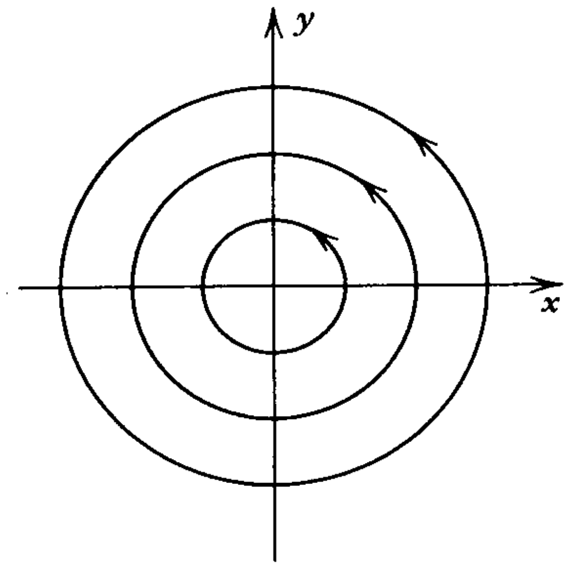
- Let us consider a uniform magnetic induction given by $\mathbf{B} = B\hat{\mathbf{z}}$ where $B = const$.



$\mathbf{A} = (0, Bx, 0)$



$\mathbf{A} = (-By, 0, 0)$



$\mathbf{A} = (-By/2, Bx/2, 0)$

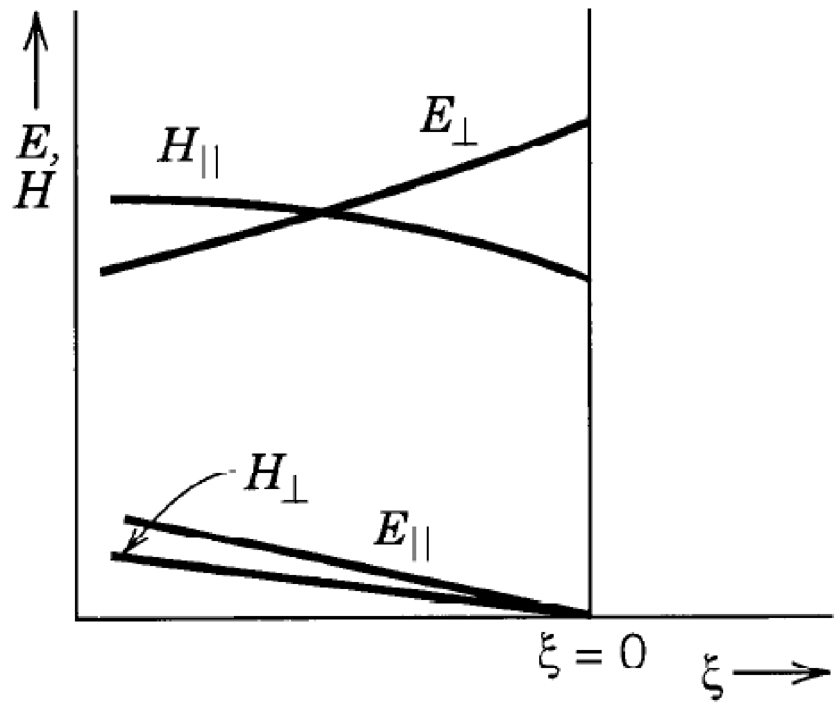
- For all three cases:

$$\nabla \cdot \mathbf{A} = 0, \quad B\hat{z} = \nabla \times \mathbf{A}$$

Boundary Conditions

- Inside perfect conductor, all of the field vectors will be zero. If $\hat{\mathbf{n}}$ is the unit normal vector pointing outward from the surface of the conductor,

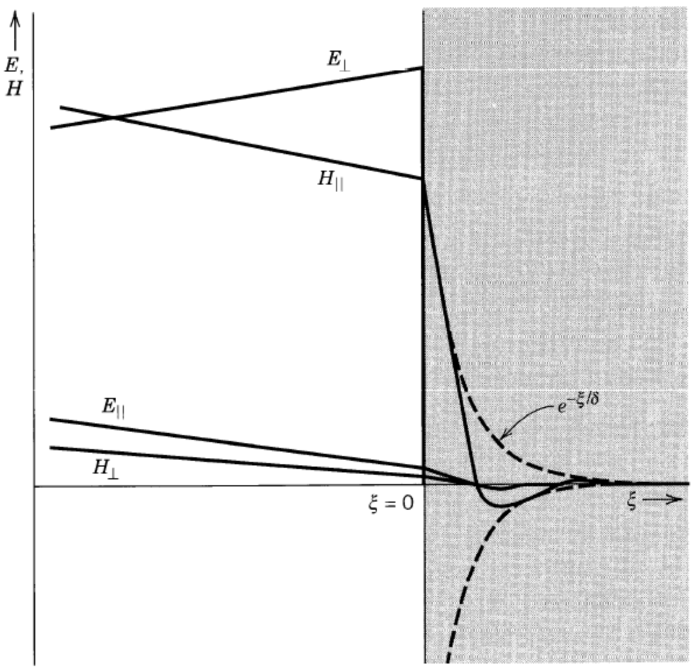
$$\hat{\mathbf{n}} \cdot \mathbf{D} = \Sigma, \quad \hat{\mathbf{n}} \times \mathbf{E} = 0, \quad \hat{\mathbf{n}} \cdot \mathbf{B} = 0, \quad \hat{\mathbf{n}} \times \mathbf{H} = \mathbf{K}$$



Skin Depth

- For a good but not perfect conductor, fields and currents are not exactly zero inside the conductor, but are confined to within a small finite layer at the surface, called the skin depth.

$$\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2}$$



- Fields inside the conductor exhibit rapid exponential decay, phase difference, magnetic field much larger than the electric field, and fields parallel to the surface.

$$\begin{aligned}\mathbf{H}_c &= \mathbf{H}_{||} e^{-\xi/\delta} e^{i\xi/\delta} \\ \mathbf{E}_c &\simeq \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)(\mathbf{n} \times \mathbf{H}_{||}) e^{-\xi/\delta} e^{i\xi/\delta}\end{aligned}$$

- Time-averaged power absorbed per unit area:

$$\frac{dP_{\text{loss}}}{da} = -\frac{1}{2}\text{Re}\left[\mathbf{n} \cdot \mathbf{E} \times \mathbf{H}^*\right] = \frac{\mu_c \omega \delta}{4}|\mathbf{H}_{||}|^2 = \frac{1}{2\sigma\delta}|\mathbf{K}_{\text{eff}}|^2$$

Energy Balance

- The energy density of the field (energy per unit volume) is*

$$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) = \frac{\epsilon_0}{2}\left(E^2 + c^2 B^2\right)$$

- The Poynting vector gives energy flow (energy per unit area per unit time) in the electromagnetic field.

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$

- Time rate of change of electromagnetic energy within a certain volume plus the energy flowing out through the boundary surface of the volume per unit time, is equal to the negative of the total work done by the fields on the sources within the volume:

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}$$

$$\frac{\partial}{\partial t}\int_V u dv + \oint_S \mathbf{S}\cdot d\mathbf{a} = -\int_V \mathbf{J}\cdot\mathbf{E}\, dv$$

*Note that for plane EM wave in vacuum: $E = cB$

Time-Harmonic Fields

- We assume all fields and sources have a time dependence $e^{-i\omega t}$ (or $e^{j\omega t}$)

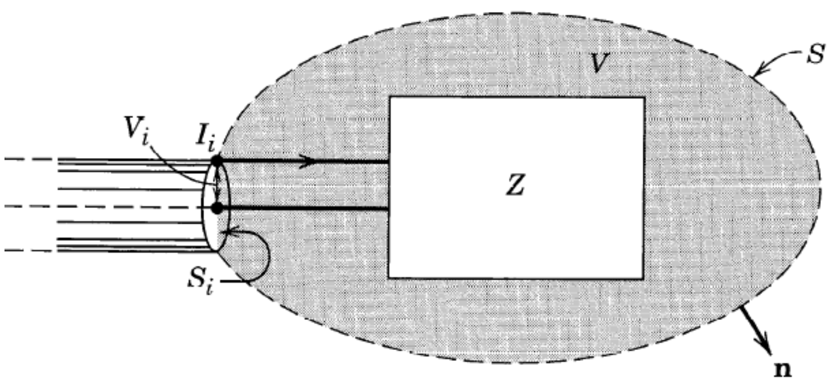
$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r})e^{-i\omega t}\right]$$

- Time-average of the products:

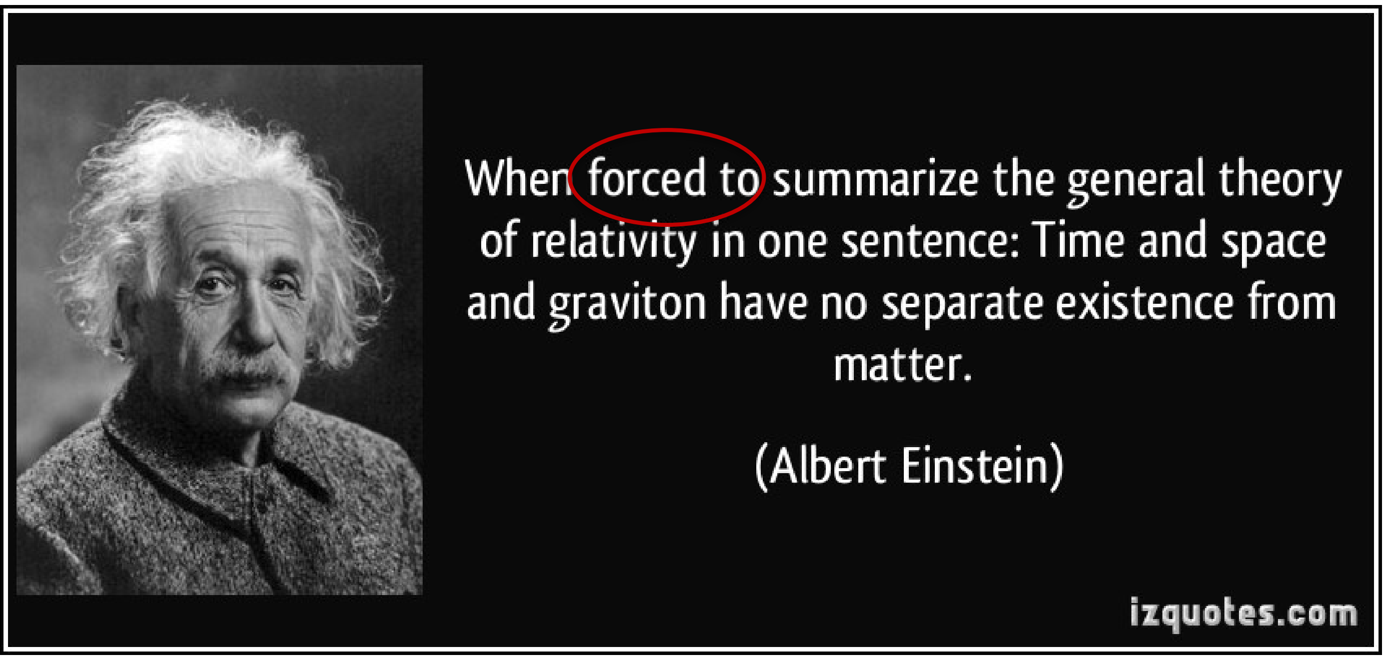
$$\langle \mathbf{J}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{J}^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})\right], \qquad \langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\right]$$

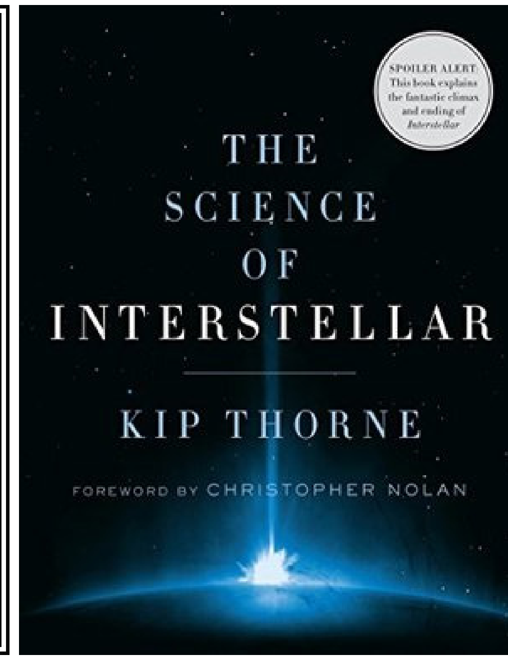
- Complex Poynting theorem:

$$\begin{aligned}\frac{1}{2}I_i^*V_i = \frac{1}{2}\underbrace{(R-iX)}_{=Z}|I_i|^2 &= -\oint_{S_i}\mathbf{S}\cdot\mathbf{n}\,da \\ &= \frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}\,dv + 2i\omega\int_V (w_e - w_m)\,dv + \oint_{S-S_i}\mathbf{S}\cdot\mathbf{n}\,da\end{aligned}$$

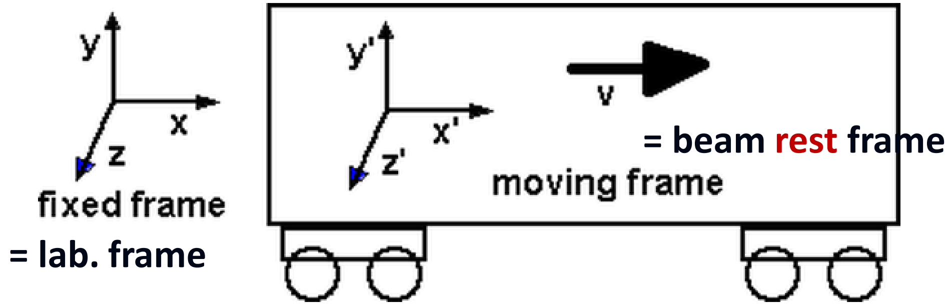


Relativity


When forced to summarize the general theory of relativity in one sentence: Time and space and graviton have no separate existence from matter.
(Albert Einstein)
izquotes.com


SPOILER ALERT
This book explains the fantastic climax and ending of Interstellar
THE SCIENCE OF INTERSTELLAR
KIP THORNE
FOREWORD BY CHRISTOPHER NOLAN

Inertial Frame



$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

- All inertial frames are in a state of constant, rectilinear motion with respect to one another; $\boldsymbol{F} = \boldsymbol{F}'$
- A non-inertial reference frame is a frame of reference that is undergoing acceleration with respect to an inertial frame.
- $\boldsymbol{F} = m\boldsymbol{a}$ holds in any coordinate system provided the term 'force' is redefined to include the so-called inertial forces.
- Lorentz transformation for relativistic motions ($\gamma = 1/\sqrt{1 - v^2/c^2} > 1$)

$$
\begin{aligned}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma\left(t - \frac{v}{c^2}x\right)
\end{aligned}
$$

Time Dilation and Lorentz Contraction

- Time interval appears to be longer to the moving observer than it does to the one at rest with respect to the clock.

$$\Delta t' = t_2' - t_1' = \gamma \left[(t_2 - t_1) - \frac{v}{c^2} \underbrace{(x_2 - x_1)}_{=0} \right] = \gamma \Delta t > \Delta t$$

Ex] Unstable particles such as muons should have a longer lifetime than resting ones as accelerated.

- As found by the moving observer, the length (whose ends are determined simultaneously) in the direction of motion will be contracted.

$$\Delta x' = x_2' - x_1' = \gamma \left[(x_2 - x_1) - v(t_2 - t_1) \right]$$

$$0 = \Delta t' = t_2' - t_1' = \gamma \left[(t_2 - t_1) - \frac{v}{c^2} (x_2 - x_1) \right]$$

$$\Delta x' = \gamma \left[1 - v^2/c^2 \right] (x_2 - x_1) = \frac{\Delta x}{\gamma} < \Delta x$$

Ex] Longitudinal Lorentz contraction of the bunch in relativistic beams.

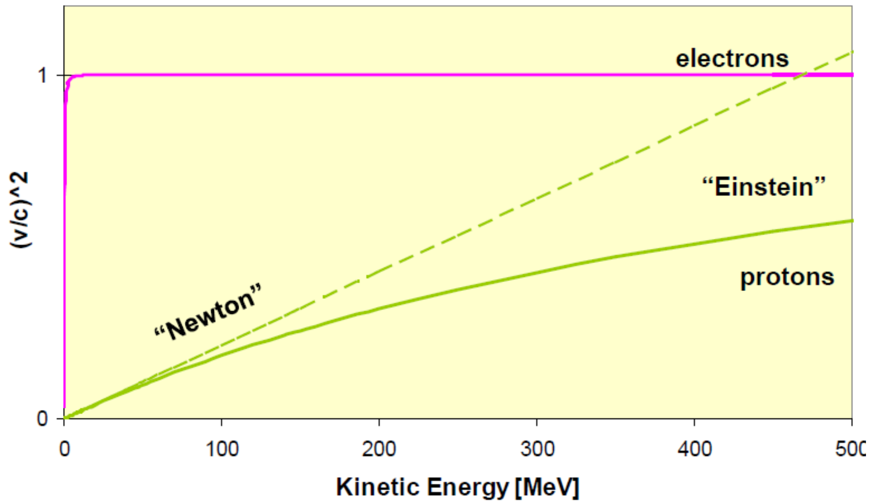
Special Relativity Handle Acceleration?

- It is often said, erroneously, that special relativity cannot deal with accelerations because it deals only with inertial frames.
- Sometimes it is claimed that general relativity is required for these situations; if that's the case, accelerator physics must have been extremely complicated!
- This is not true. We must, of course, only allow transformations between inertial frames; the frames must not accelerate.
- Special relativity treats acceleration differently from inertial frames and can deal with anything kinematic, but general relativity is required when gravitational forces are present.

Main Results of Special Relativity

- Relativistic parameters: Don't be confused with Twiss parameters.

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1 + \frac{1}{2}\beta^2, \quad \beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$$



- The total energy, mechanical momentum, and kinetic energy of a rest mass m:

$$U = \gamma mc^2, \quad p = \gamma m\beta c, \quad W = (\gamma - 1)mc^2 \longrightarrow \frac{1}{2}mv^2$$

- The relation between total energy and momentum in the absence of EM fields:

$$U = \sqrt{p^2c^2 + (mc^2)^2}, \quad \text{or} \quad \gamma^2 = (\beta\gamma)^2 + 1$$

Energy and Mass Units

- To describe the energy of individual particles, we use the eV, the energy that a unit charge

$$e = 1.6 \times 10^{-19} \text{ Coulomb}$$

gains when it falls through a potential, $\Delta\phi$ = 1 volt.

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ Joule}$$

- We can use Einstein's relation to convert rest mass to energy units.

$$E = mc^2$$

- For electrons,

$$E = (9.11 \times 10^{-31} \text{ kg}) \times (3 \times 10^8 \text{ m/s})^2 = 0.512 \text{ MeV}$$

- For protons,

$$E = (1.67 \times 10^{-27} \text{ kg}) \times (3 \times 10^8 \text{ m/s})^2 = 938 \text{ MeV}$$

Lorentz Equation and Effective Mass

- Lorentz equation: We need to consider changes of γ in time.

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m \mathbf{v}) = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\frac{d\gamma}{dt} = \beta(1-\beta^2)^{-3/2}\frac{d\beta}{dt} = \frac{\beta\gamma^3}{c}\frac{dv}{dt}$$

- Parallel and perpendicular decomposition:

$$\gamma m \left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} + \gamma m \left(\frac{d\mathbf{v}}{dt}\right)_{\perp} + m\mathbf{v}\frac{d\gamma}{dt} = \mathbf{F}_{\perp} + \mathbf{F}_{\parallel}$$

  – Parallel acceleration:

$$\gamma m \left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} + \gamma m \beta^2\gamma^2 \left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} = \mathbf{F}_{\parallel} \longrightarrow \underbrace{\gamma^3 m}_{\text{longitudinal mass}} \left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} = \mathbf{F}_{\parallel}$$

  – Perpendicular acceleration:

$$\gamma m \left(\frac{d\mathbf{v}}{dt}\right)_{\perp} = \mathbf{F}_{\perp} \longrightarrow \underbrace{\gamma m}_{\text{perpendicular mass}} \left(\frac{d\mathbf{v}}{dt}\right)_{\perp} = \mathbf{F}_{\perp}$$

Transformation of Potentials and Fields

- Lorentz transformation of potentials:

$$A'_x = \gamma\left(A_x - \frac{v}{c^2}\phi\right), \quad A'_y = A_y, \quad A'_z = A_z, \quad \phi' = \gamma\left(\phi - vA_x\right)$$

- Lorentz transformation of fields: Longitudinal fields are "Lorentz invariant"

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}_{\perp}\right), \quad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2}\times\mathbf{E}_{\perp}\right)$$

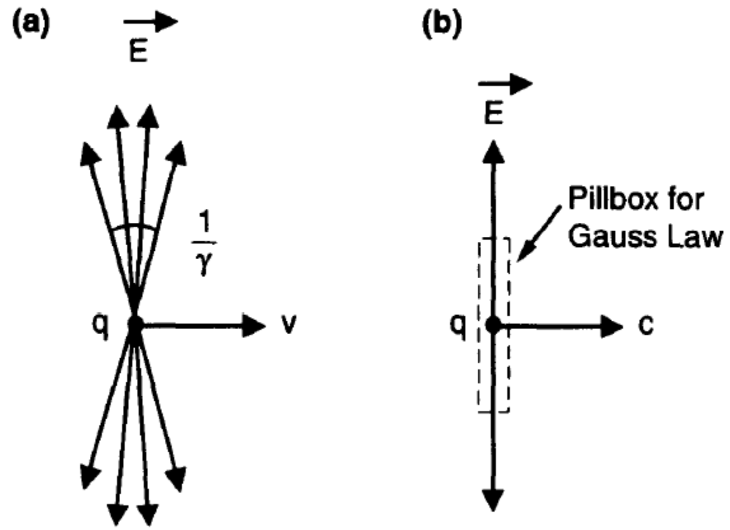
Ex] Pure electric field in beam rest frame (i.e., primed system): A pure electric field to one observer may be seen as both an electric and a magnetic field to a second observer.

$$\mathbf{E}_{\perp} = \gamma\left(\mathbf{E}'_{\perp} - \mathbf{v}\times\cancel{\mathbf{B}'_{\perp}}\right), \quad \mathbf{B}_{\perp} = \gamma\left(\cancel{\mathbf{B}'_{\perp}} + \frac{\mathbf{v}}{c^2}\times\mathbf{E}'_{\perp}\right)$$

$$\mathbf{F}_{\perp} = q\left(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}_{\perp}\right) = \frac{q\mathbf{E}_{\perp}}{\gamma^2}$$

Fields of Relativistic Point Charge

- The field distribution is Lorentz contracted into a thin disk perpendicular to the particle's direction of motion with an angular spread on the order of $1/\gamma$.



$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{q}{\epsilon_0}\delta[\mathbf{r} - \mathbf{r}_0(t)] = \frac{q}{\epsilon_0}\frac{\delta(r)}{2\pi r}\delta(z - ct)$$

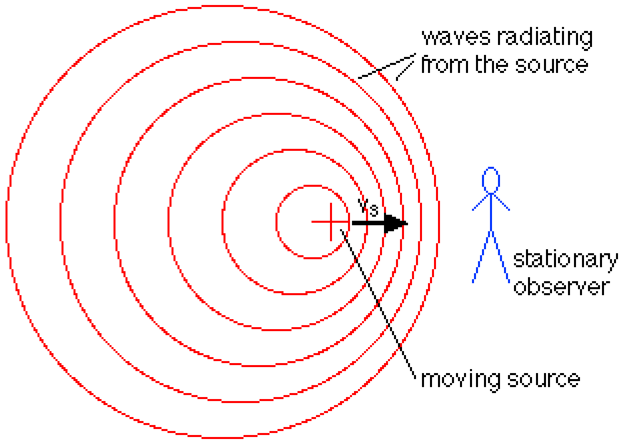
$$\longrightarrow \int 2\pi r E_r dz = \int\int \frac{q}{\epsilon_0}\frac{\delta(r)}{\cancel{2\pi r}}\delta(z-ct)\cancel{2\pi r}drdz \longrightarrow E_r = \frac{q}{2\pi\epsilon_0 r}\delta(z-ct)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0 q \mathbf{v}\delta[\mathbf{r} - \mathbf{r}_0(t)] = \mu_0 q \mathbf{v}\frac{\delta(r)}{2\pi r}\delta(z-ct)$$

$$\longrightarrow 2\pi r B_\theta = \int \mu_0 q c \frac{\delta(r)}{\cancel{2\pi r}}\delta(z-ct)\cancel{2\pi r}dr \longrightarrow B_\theta = \frac{\mu_0 q}{2\pi r}c\delta(z-ct) = \frac{E_r}{c}$$

Relativistic Doppler Shift

- The Doppler effect is modified to be consistent with the Lorentz transformation

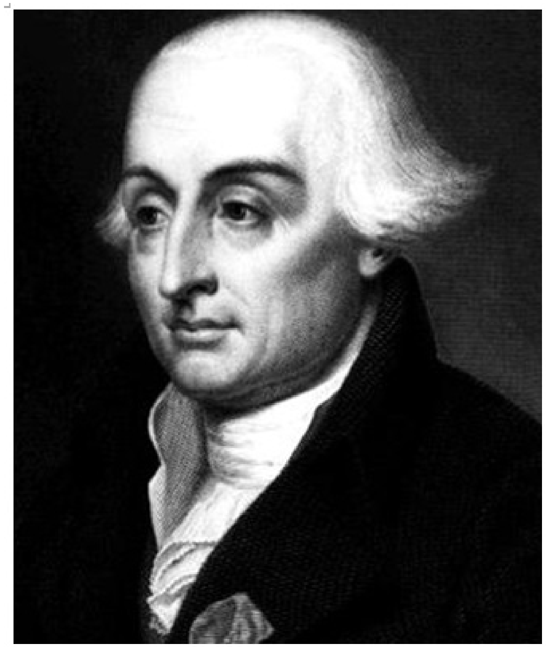


$$\frac{\lambda_{observed}}{\lambda_{source}} = \frac{f_{source}}{f_{observed}} = \sqrt{\frac{1-\beta}{1+\beta}} \longrightarrow \sqrt{\frac{1/2\gamma^2}{2}} = \frac{1}{2\gamma}$$

Ex] Fundamental radiation wavelength from undulator, which is a periodic arrangement (λ_u) of many short dipole magnets of alternating polarity.

- Electron sees length contraction of the undulator period: $\lambda_u' = \lambda_u/\gamma$
- The electrons oscillate at a corresponding higher frequency: $\omega = 2\pi c/\lambda_u'$
- The electrons emit radiation just like an oscillation dipole: $P = (e^2/6\pi\epsilon_0 c^3)\dot{\mathbf{v}}^2 \propto \omega^4$
- For a stationary observer looking against the electron beam, the radiation appears strongly blue-shifted: $\lambda_{observed} \approx \lambda_u'/(2\gamma) \approx \lambda_u/(2\gamma^2)$

Classical Mechanics



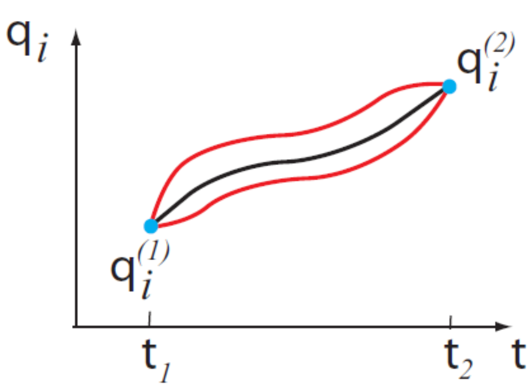
Lagrangian Mechanics

- If we take the nonrelativistic case of a conservative system and **B** = 0, the Lagrange function is defined by the difference between kinetic and potential energy.

$$L = T - V$$

- Hamilton's variational principle states that the motion of the system from one fixed point at time t_1 to another point at time t_2 is such that the time integral of the Lagrangian along the path taken is an extremum (actually, a minimum).

$$\delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)\ dt = \int_{t_1}^{t_2} \delta L(\mathbf{q}, \dot{\mathbf{q}}, t)\ dt = 0$$



- Lagrangian equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- Canonical momenta (or conjugate momenta):

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Hamiltonian Mechanics

- Hamiltonian is constructed from a Lagrangian:

$$H(\mathbf{q}, \mathbf{p}, t) = \mathbf{p} \cdot \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

- Hamiltonian equations of motion:

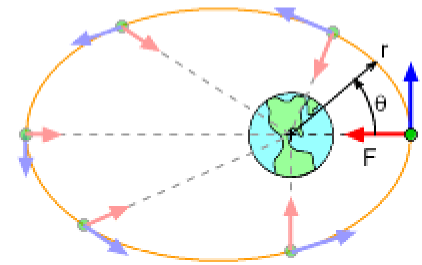
$$\begin{aligned} dH &= \sum_i \left[\left(\frac{\partial H}{\partial q_i} \right) dq_i + \left(\frac{\partial H}{\partial p_i} \right) dp_i \right] + \left(\frac{\partial H}{\partial t} \right) dt \\ &= \sum_i \left[\dot{q}_i dp_i + p_i d\dot{q}_i - \underbrace{\left(\frac{\partial L}{\partial q_i} \right)}_{=\dot{p}_i} dq_i - \underbrace{\left(\frac{\partial L}{\partial \dot{q}_i} \right)}_{=p_i} d\dot{q}_i \right] - \left(\frac{\partial L}{\partial t} \right) dt \end{aligned}$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

- Conservation of Hamiltonian: if it does not depend explicitly on t.

$$\begin{aligned} \frac{dH}{dt} &= \sum_i \left[\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right] + \frac{\partial H}{\partial t} \\ &= \sum_i \left[-\cancel{\dot{p}_i \dot{q}_i} + \cancel{\dot{p}_i \dot{q}_i} \right] + \frac{\partial H}{\partial t} \end{aligned}$$

Example



- Lagrangian for a central force problem in 2D:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

- Canonical momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

- Hamiltonian:

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

- Equations of motion:

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \frac{dp_\theta}{dt} = -\frac{\partial H}{\partial \theta} = 0 \longrightarrow p_\theta = const. \equiv l$$

$$\frac{dr}{dt} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{l^2}{mr^3} - \frac{\partial V(r)}{\partial r}$$

Relativistic Dynamics in EM Fields

- Lagrangian with velocity-independent potentials:

$$L = mc^2\left[1-\left(1-\beta^2\right)^{1/2}\right] - q\phi + q\mathbf{v}\cdot\mathbf{A} \xrightarrow{\text{neglect const.}} L = -mc^2\left(1-v^2/c^2\right)^{1/2} - q\phi + q\mathbf{v}\cdot\mathbf{A}$$

- Canonical momenta: mechanical momenta + vector potential contribution

$$\mathbf{v} = (\dot{x}, \dot{y}, \dot{z}) \quad \text{or} \quad (\dot{r}, r\dot{\theta}, \dot{z})$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Ex] Cartesian coordinates:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \gamma m \dot{x} + qA_x$$

Ex] Cylindrical coordinates:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \gamma m r^2 \dot{\theta} + qrA_\theta$$

Relativistic Dynamics in EM Fields

- Hamiltonian: Using cartesian coordinates, one can prove

$$\begin{aligned} H &= \sum_i p_i \dot{q}_i - L \\ &= \mathbf{v}\cdot(\gamma m\mathbf{v} + q\mathbf{A}) + mc^2\sqrt{1 - v^2/c^2} - q\mathbf{v}\cdot\mathbf{A} + q\phi \\ &= \gamma mc^2\left(\beta^2 + \frac{1}{\gamma^2}\right) + q\phi = \gamma mc^2 + q\phi \\ &= \sqrt{\gamma^2 m^2 v^2 c^2 + m^2 c^4} + q\phi = \sqrt{p_{mech}^2 c^2 + m^2 c^4} + q\phi \end{aligned}$$

Ex] Cartesian coordinates:

$$\begin{aligned} H &= c\left[(\mathbf{p} - q\mathbf{A})^2 + m^2c^2\right]^{1/2} + q\phi \\ &= c\left[(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2 + m^2c^2\right]^{1/2} + q\phi \end{aligned}$$

Ex] 4-vectors:

$$\left(\mathbf{p} - q\mathbf{A}, \frac{i(H - q\phi)}{c}\right) \longrightarrow (\mathbf{p} - q\mathbf{A})^2 - \frac{(H - q\phi)^2}{c^2} = -m^2c^2$$

Ex] Cylindrical coordinates:

$$H = c\left[(p_r - qA_r)^2 + \left(\frac{p_\theta - qrA_\theta}{r}\right)^2 + (p_z - qA_z)^2 + m^2c^2\right]^{1/2} + q\phi$$

Canonical Transformation

- The variation of the action integral between two fixed endpoints:

$$\delta \int_{t_1}^{t_2} L\ dt = \delta \int_{t_1}^{t_2} [\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t)]\ dt = 0$$

- We would like to transform from the old coordinate system $(\mathbf{q}, \mathbf{p})$ to a new system $(\mathbf{Q}, \mathbf{P})$ with a new Hamiltonian $K(\mathbf{Q}, \mathbf{P}, t)$:

$$\delta \int_{t_1}^{t_2} [\mathbf{P} \cdot \dot{\mathbf{Q}} - K(\mathbf{Q}, \mathbf{P}, t)]\ dt = 0$$

- One way for both vibrational integral equalities to be satisfied is to have

$$\lambda[\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t)] = \mathbf{P} \cdot \dot{\mathbf{Q}} - K(\mathbf{Q}, \mathbf{P}, t) + \frac{dF}{dt}$$

- If $\lambda \neq 1$, it is extended canonical transformation. If $\lambda \neq 1$ and $\frac{dF}{dt} = 0$, it is scale transformation. These transformations do not preserve phase space volume

Generating Function

- The function F is in general a function of both the old and new variables as well as the time. We will restrict ourselves to functions that contain half of the old variables and half the new; these are useful for determining the explicit form of the transformation.

Case 1:
$$F = F_1(\mathbf{q}, \mathbf{Q}, t) \qquad p_i = +\frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$$

Case 2:
$$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q}\cdot\mathbf{P} \qquad p_i = +\frac{\partial F_2}{\partial q_i}, \quad Q_i = +\frac{\partial F_2}{\partial P_i}$$

Case 3:
$$F = F_3(\mathbf{Q}, \mathbf{p}, t) + \mathbf{q}\cdot\mathbf{p} \qquad q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

Case 4:
$$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q}\cdot\mathbf{p} - \mathbf{Q}\cdot\mathbf{P} \qquad q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = +\frac{\partial F_4}{\partial P_i}$$

- In all cases, new Hamailtonan and equations of motion become:

$$K = H + \frac{\partial F_i}{\partial t}, \qquad \frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i}, \qquad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i}$$

Example

- For F_3 we will show

$$\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t) = \mathbf{P}\cdot\dot{\mathbf{Q}} - K(\mathbf{Q},\mathbf{P},t) + \frac{dF}{dt}$$

- Proof:

$$F = F_3(\mathbf{Q},\mathbf{p},t) + \mathbf{q}\cdot\mathbf{p}$$

$$\begin{aligned}\frac{dF}{dt} &= \frac{\partial F_3}{\partial t} + \sum_i \left(\underbrace{\frac{\partial F_3}{\partial Q_i}}_{=-P_i} \dot{Q}_i + \underbrace{\frac{\partial F_3}{\partial p_i}}_{=-q_i} \dot{p}_i + +q_i\dot{p}_i + p_i\dot{q}_i \right) \\ &= \frac{\partial F_3}{\partial t} + \sum_i \left(-P_i\dot{Q}_i - \cancel{q_i\dot{p}_i} + \cancel{q_i\dot{p}_i} + p_i\dot{q}_i \right) \\ &= \frac{\partial F_3}{\partial t} - \mathbf{P}\cdot\dot{\mathbf{Q}} + \mathbf{p}\cdot\dot{\mathbf{q}}\end{aligned}$$

Therefore

$$K = H + \frac{\partial F_3}{\partial t} \longrightarrow \mathbf{p}\cdot\dot{\mathbf{q}} - H = \mathbf{P}\cdot\dot{\mathbf{Q}} - K + \frac{dF}{dt}$$

Change the Role of Time Coordinates

- Provided that the reference particle moves without backtracking, or some particle coordinate increases in time, we can change the role of that coordinate and time.

$$\int_{t_1}^{t_2} [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]\,dt = \int_{t_1}^{t_2} [\mathbf{p}\cdot d\mathbf{q} - H(\mathbf{q},\mathbf{p},t)dt]$$

$$\mathbf{p}\cdot d\mathbf{q} - Hdt = \sum_i p_i dq_i + (-H)dt = \left(\sum_{i\neq j} p_i dq_i + (-H)dt\right) - (-p_j)dq_j$$

$$\begin{aligned} H &\longleftrightarrow -p_j \\ t &\longleftrightarrow q_j \end{aligned}$$

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \longleftrightarrow \frac{dt}{dq_j} = \frac{\partial(-p_j)}{\partial(-H)}, \quad \frac{d(-H)}{dq_j} = -\frac{\partial(-p_j)}{\partial t}$$

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*Special thanks to USPAS director, Prof. William Barletta, who provides the USPAS lecture slides and allows me to reuse some of them for this lecture.