Lecture 4
Longitudinal Dynamics

(Ch. 3 & 4 of FOBP, Ch. 2 & 5 of UP-ALP)

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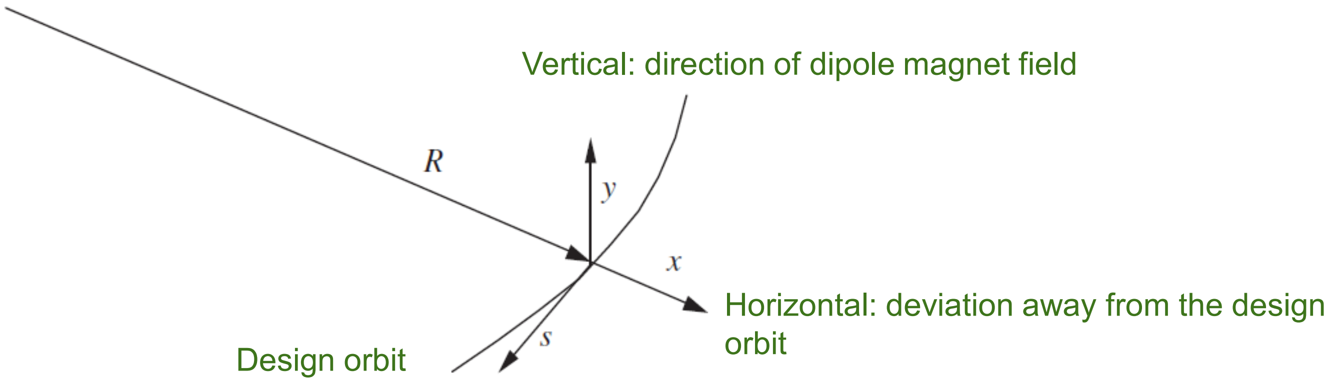
Off-Momentum Effects

(Dispersion, Momentum compaction, and Chromaticity)

[Review]

- Equation of motion in the horizontal direction with correct momentum:



$$ds = Rd\phi$$

$$x'' + \left[\left(\frac{1}{R_0}\right)^2 + \frac{qB'}{p_0}\right] x = 0, \qquad B_0 R_0 = \frac{p_0}{q}$$

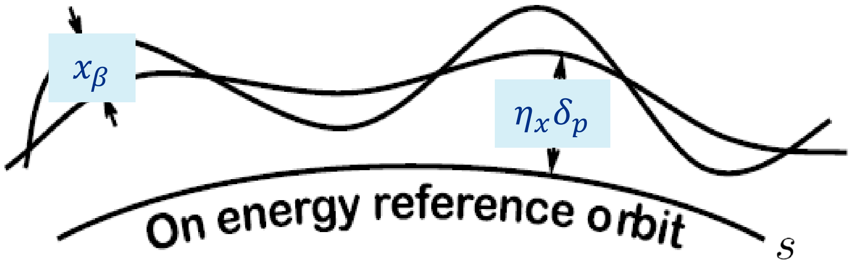
From dipole components — From quadrupole components — Magnetic rigidity

Dispersion (η or D)

- Change in the design orbit for the off-momentum particle:

$$x = x_\beta + \eta_x \frac{(p - p_0)}{p_0} = x_\beta + \eta_x \frac{\Delta p}{p_0} = x_\beta + \eta_x \delta_p$$

Offset in position   Offset in momentum



- In Lecture 2, we used the following force balance equation for constant v_0:

$$\frac{dp_x}{dt} = \frac{\gamma m_0 v_0^2}{R_0(1 + x/R_0)} - qv_0 B_0 \simeq \frac{\gamma m_0 v_0^2}{R_0}(1 - x/R_0 + \cdots) - qv_0 B_0$$

$$\simeq -\frac{\gamma m_0 v_0^2}{R_0^2}x + \frac{\gamma m_0 v_0^2}{R_0} - qv_0 B_0$$

$$B_0 R_0 = \frac{p_0}{q}$$

Now we allow v to be deviated from v_0

$$\frac{dp_x}{dt} = \frac{\gamma m_0 v^2}{R_0(1 + x/R_0)} - qv B_0 \simeq \frac{\gamma m_0 v^2}{R_0}(1 - x/R_0 + \cdots) - qv B_0$$

$$\simeq -\frac{\gamma m_0 v^2}{R_0^2}x + \frac{\gamma m_0 v^2}{R_0} - qv B_0$$

$$\simeq -\frac{\gamma m_0 v^2}{R_0^2}x + \gamma m_0 v^2 \left[\frac{1}{R_0} - \frac{qB}{p}\right]$$

Path length focusing term   New term caused by $p \neq p_0$

Governing equation for dispersion

- We can express the new force balance equation using s as an independent variable:

$$(\cdots)' \equiv \frac{d}{ds} = \frac{d}{vdt}, \quad p_x = \gamma m_0 \frac{dx}{dt}$$

$$x'' = -\frac{1}{R_0^2}x + \left[\frac{1}{R_0} - \frac{qB_0}{p}\right] \simeq -\frac{1}{R_0^2}x + \left[\frac{1}{R_0} - \frac{1}{R_0}\left(1 - \frac{\Delta p}{p_0}\right)\right] = -\frac{1}{R_0^2}x + \frac{1}{R_0}\frac{\Delta p}{p_0}$$

$$\frac{pB_0}{p} = \frac{1}{R_0}\frac{p_0}{p} = \frac{1}{R_0}\frac{p_0}{p_0 + \Delta p} \simeq \frac{1}{R_0}\left(1 - \frac{\Delta p}{p_0}\right)$$

- With the quadrupole term included,

$$x'' + \left[\frac{1}{R_0^2} + \frac{qB'}{p_0}\right]x = \frac{1}{R_0}\frac{\Delta p}{p_0}$$

1st order in position offset

1st order in momentum offset

- If we substitute $x = x_\beta + \eta_x \frac{\Delta p}{p_0}$

$$x_\beta'' + \left[\frac{1}{R_0^2} + \frac{qB'}{p_0}\right]x_\beta = 0, \quad \eta_x''\frac{\Delta p}{p_0} + \left[\frac{1}{R_0^2} + \frac{qB'}{p_0}\right]\eta_x\frac{\Delta p}{p_0} = \frac{1}{R_0}\frac{\Delta p}{p_0}$$

$$\eta_x'' + \underbrace{\left[\frac{1}{R_0^2} + \frac{qB'}{p_0}\right]}_{\equiv \kappa_b^2}\eta_x = \frac{1}{R_0}$$

Solution of the dispersion equation

- For net horizontal focusing, the general solution is composed of homogeneous and particular solutions:

$$\eta_x = \underbrace{A\cos(\kappa_b s) + B\sin(\kappa_b s)}_{\text{homogeneous}} + \underbrace{\frac{1}{\kappa_b^2 R_0}}_{\text{particular}}$$

[Note] If there is only bending magnet (i.e., $B' = 0$, no quadrupole),

$$\eta_{x,part} = \frac{1}{\kappa_b^2 R_0} = R_0$$

- If we apply matching boundary conditions at the entrance of the bend magnet ($s = 0$),

$$\eta_x(s) = \left[\eta_x(0) - \frac{1}{\kappa_b^2 R_0}\right]\cos(\kappa_b s) + \frac{\eta_x'(0)}{\kappa_b}\sin(\kappa_b s) + \frac{1}{\kappa_b^2 R_0}$$

$$\eta_x'(s) = \left[\frac{1}{\kappa_b R_0} - \kappa_b \eta_x(0)\right]\sin(\kappa_b s) + \eta_x'(0)\cos(\kappa_b s)$$

Transfer matrix of dispersion

- In the matrix form,

$$\begin{bmatrix} \eta_x(s) \\ \eta_x'(s) \\ 1 \end{bmatrix} = \begin{bmatrix} \cos[\kappa_b s] & \frac{1}{\kappa_b}\sin[\kappa_b s] & \frac{1-\cos(\kappa_b s)}{\kappa_b^2 R_0} \\ -\kappa_b \sin[\kappa_b s] & \cos[\kappa_b s] & \frac{\sin(\kappa_b s)}{\kappa_b R_0} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_x(0) \\ \eta_x'(0) \\ 1 \end{bmatrix}$$

[Note]

1. Even if there is no dispersion in the beginning (i.e., $\eta_x(0) = \eta_x'(0) = 0$), dispersion can be created when the beam is transported through a bending magnet.
2. In a straight section ($R_0 \to \infty$, i.e., no bending),

$$\begin{bmatrix} \eta_x(s) \\ \eta_x'(s) \\ 1 \end{bmatrix} = \begin{bmatrix} \cos[\kappa_b s] & \frac{1}{\kappa_b}\sin[\kappa_b s] & 0 \\ -\kappa_b \sin[\kappa_b s] & \cos[\kappa_b s] & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_x(0) \\ \eta_x'(0) \\ 1 \end{bmatrix}$$

3. Even in the straight section, dispersion can exist if there is dispersion in the beginning (i.e., $\eta_x(0) \neq 0, \eta_x'(0) \neq 0$).

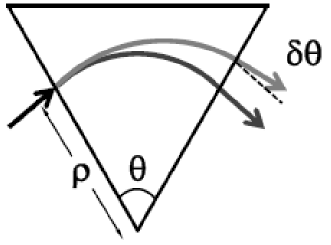


FIGURE 2.18
Bending magnet creates dispersion.

Longitudinal coordinate

- The canonical dependent coordinate in the longitudinal direction is time of arrival relative to the design particle.

$$\tau = t - t_0$$

Early particle: < 0

Late particle: > 0

- In the Hamiltonian analysis, it is useful to introduce a parametrization of the time through a spatial variable,

$$\zeta = -v_0 \tau = v_0 t_0 - v_0 t = s - v_0 t = s - \beta_0 c t$$

Early particle: > 0

Late particle: < 0

[Note] This is the distance that must be traveled at the design velocity by the design particle, to reach the position of the temporally advanced (or delated) particle.

Momentum compaction

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- The time of flight of an off-momentum particle through travel distance $L(p)$:

$$t(p) = \frac{L(p)}{v(p)}$$

- First order expansion with paraxial approximation yields

$$\delta t = \delta \tau = \frac{\delta L}{v_z} - \frac{L}{v_z^2}\delta v_z \simeq \frac{\delta L}{v_0} - \frac{L_0}{v_0^2}\delta v_z$$

$$\frac{\delta \tau}{t_0} = \frac{\delta L}{L_0} - \frac{\delta v_z}{v_0} \simeq \left[\alpha_c - \frac{1}{\gamma_0^2}\right]\frac{\delta p}{p_0} \qquad t_0 = \frac{L_0}{v_0}$$

Here we define the path length parameter (usually called, momentum compaction) as

$$\alpha_c \equiv \frac{\delta L / L_0}{\delta p / p_0}$$

which characterizes the path length changes according to the momentum offset. We also used

$$\frac{\delta v_z}{v_0} = \frac{\delta \beta}{\beta_0} \simeq \frac{1}{\gamma_0^2}\frac{\delta p}{p_0}$$

$$\delta p = \delta(mc\gamma\beta) = mc(\beta\delta\gamma + \gamma\delta\beta)$$
$$\delta\gamma = \gamma^3\beta\delta\beta$$

Phase slip factor (or time dispersion)

- We define so-called phase slip factor:

$$\frac{\delta\tau}{t_0} \simeq \left[\alpha_c - \frac{1}{\gamma_0^2}\right]\frac{\delta p}{p_0} \equiv \eta_\tau \frac{\delta p}{p_0}$$

$$\eta_\tau \equiv \frac{\partial(\delta\tau/t_0)}{\partial(\delta p/p_0)} = \alpha_c - \frac{1}{\gamma_0^2}$$

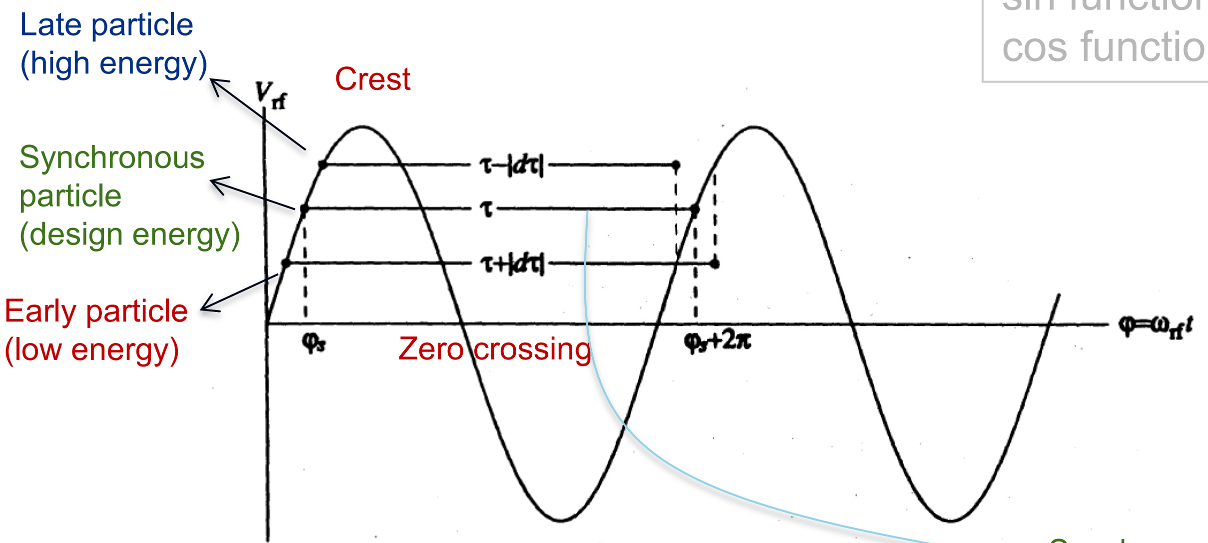
Note that there is a certain energy ($\gamma_0 = \gamma_{tr}$, called transition energy) at which the time dispersion vanishes, and all particle pass through the system in the same amount of time.

$$\eta_\tau = 0 = \alpha_c - \frac{1}{\gamma_0^2} = \alpha_c - \frac{1}{\gamma_{tr}^2}$$

- Below transition: $\eta_\tau = \frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma_0^2} < 0, \quad \gamma_0 < \gamma_{tr}$
  - Particles of higher momentum pass through the system more quickly, which is the natural state of affairs in linear systems.
- Above transition: $\eta_\tau = \frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma_0^2} > 0, \quad \gamma_0 > \gamma_{tr}$
  - Particles of higher momentum take more time to pass the system , since the added path length of a higher-momentum trajectory outweighs the added advantage in velocity, which becomes progressively smaller as particle becomes more relativistic.

[Example]

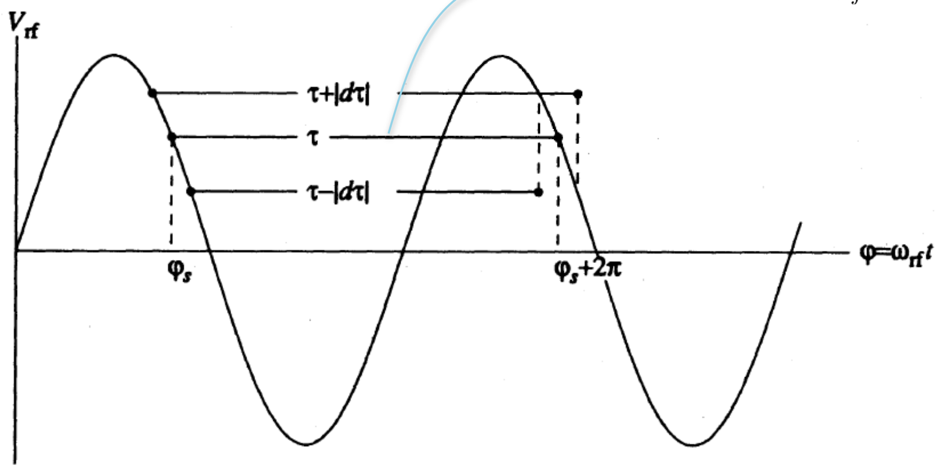
*Convention:
sin function for circular machines, and
cos function for linear machines.



Below transition: stable oscillation for off crest with $\Delta\varphi < 0$

Synchronous particle arrives at the same voltage

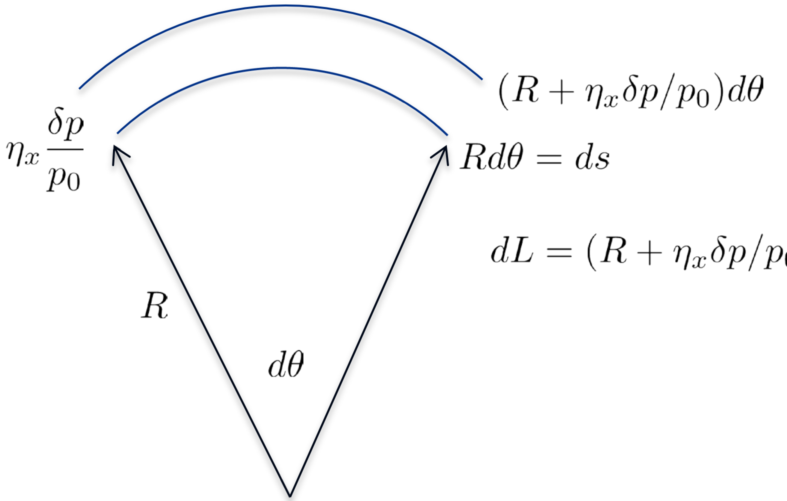
$$\omega_{rf}\tau = 2\pi, \quad \text{or} \quad 2\pi h \quad (h = 1, 2, \cdots)$$



Above transition: stable oscillation for off crest with $\Delta\varphi > 0$

Momentum compaction VS Dispersion

- Path length change around the circular path:



$$dL = (R+\eta_x\delta p/p_0)d\theta - Rd\theta = \eta_x\delta p/p_0 d\theta$$

$$\delta L = \int dL = \frac{\delta p}{p_0}\int \eta_x d\theta = \frac{\delta p}{p_0}\int \frac{\eta_x}{R}ds \equiv \delta_p R_{56}$$

- For a single pass system:

$$\alpha_c = \frac{\delta L/L_0}{\delta p/p_0} = \frac{1}{s-s_0}\int_{s_0}^{s}\frac{\eta_x(\tilde{s})}{R(\tilde{s})}d\tilde{s}$$

- For a closed system:

$$\alpha_c = \frac{\delta L/L_0}{\delta p/p_0} = \frac{1}{C_0}\oint \frac{\eta_x(\tilde{s})}{R(\tilde{s})}d\tilde{s}$$

→ Circumference of the design orbit

Chromaticity (or Chromatic aberration)

- Offsets of energy in the particles cause not only dispersion but also result in different focusing strengths of the magnetic elements:

$$\frac{qB'}{p} = \frac{qB'}{p_0(1+\delta_p)} = \frac{qB'}{p_0}(1-\delta_p) = k_0(1-\delta_p)$$

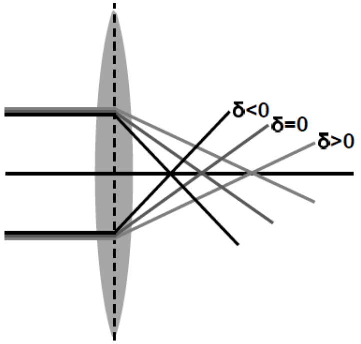
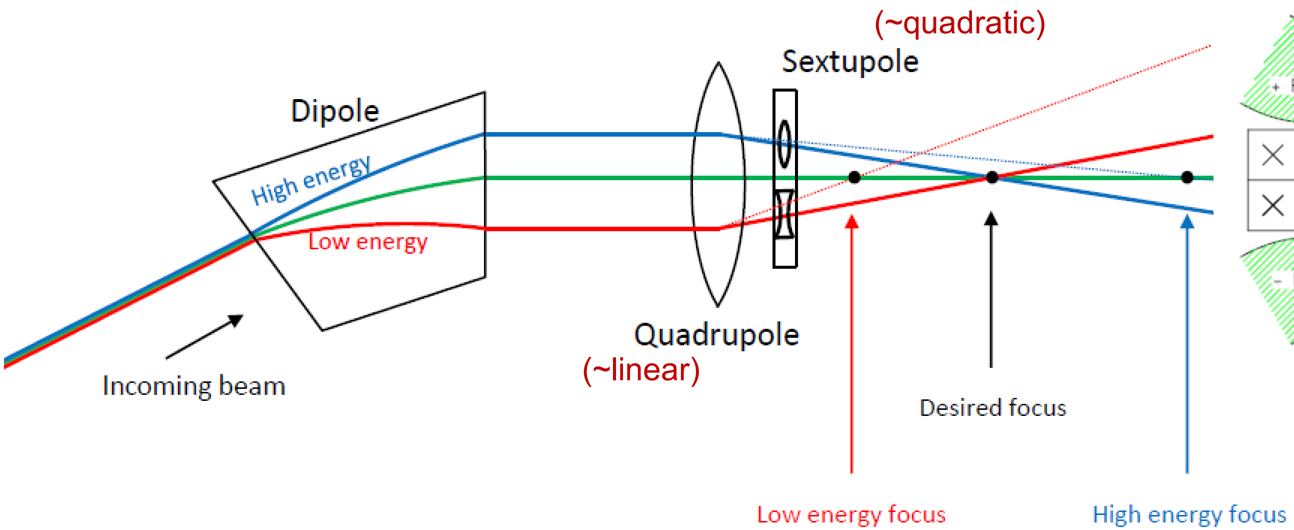
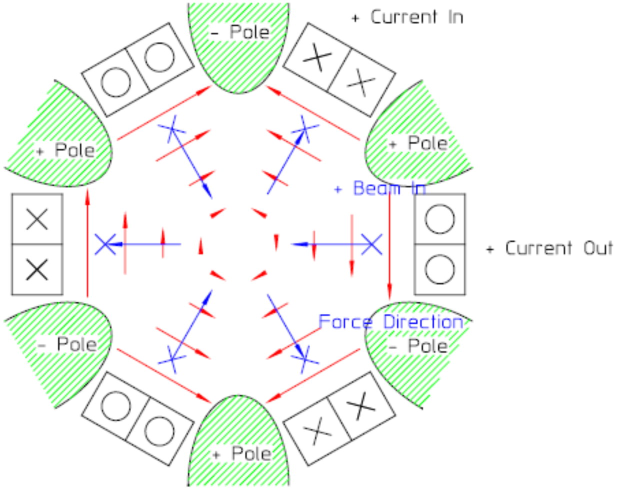


FIGURE 2.20
Chromaticity of a focusing quadrupole.





Sextupole:
→ Nonlinearity
→ Coupling

6D linear transformation

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- 6D phase space vector:

$$\Phi = (x, x', y, y', \zeta = s - v_0 t, \zeta' \simeq \delta p/p_0 = \delta_p)^T$$

$$\Phi(s) = \mathbf{R}(s, s_0) \cdot \Phi(s_0)$$

$$\mathbf{R}(s,s_0) = \begin{bmatrix} \frac{\partial x_f}{\partial x_i} & \frac{\partial x_f}{\partial x'_i} & \frac{\partial x_f}{\partial y_i} & \frac{\partial x_f}{\partial y'_i} & \frac{\partial x_f}{\partial \zeta_i} & \frac{\partial x_f}{\partial \zeta'_i} \\ \frac{\partial x'_f}{\partial x_i} & \frac{\partial x'_f}{\partial x'_i} & \frac{\partial x'_f}{\partial y_i} & \frac{\partial x'_f}{\partial y'_i} & \frac{\partial x'_f}{\partial \zeta_i} & \frac{\partial x'_f}{\partial \zeta'_i} \\ \frac{\partial y_f}{\partial x_i} & \frac{\partial y_f}{\partial x'_i} & \frac{\partial y_f}{\partial y_i} & \frac{\partial y_f}{\partial y'_i} & \frac{\partial y_f}{\partial \zeta_i} & \frac{\partial y_f}{\partial \zeta'_i} \\ \frac{\partial y'_f}{\partial x_i} & \frac{\partial y'_f}{\partial x'_i} & \frac{\partial y'_f}{\partial y_i} & \frac{\partial y'_f}{\partial y'_i} & \frac{\partial y'_f}{\partial \zeta_i} & \frac{\partial y'_f}{\partial \zeta'_i} \\ \frac{\partial \zeta_f}{\partial x_i} & \frac{\partial \zeta_f}{\partial x'_i} & \frac{\partial \zeta_f}{\partial y_i} & \frac{\partial \zeta_f}{\partial y'_i} & \frac{\partial \zeta_f}{\partial \zeta_i} & \frac{\partial \zeta_f}{\partial \zeta'_i} \\ \frac{\partial \zeta'_f}{\partial x_i} & \frac{\partial \zeta'_f}{\partial x'_i} & \frac{\partial \zeta'_f}{\partial y_i} & \frac{\partial \zeta'_f}{\partial y'_i} & \frac{\partial \zeta'_f}{\partial \zeta_i} & \frac{\partial \zeta'_f}{\partial \zeta'_i} \end{bmatrix}$$

$$= \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{bmatrix}$$

η_x, η'_x

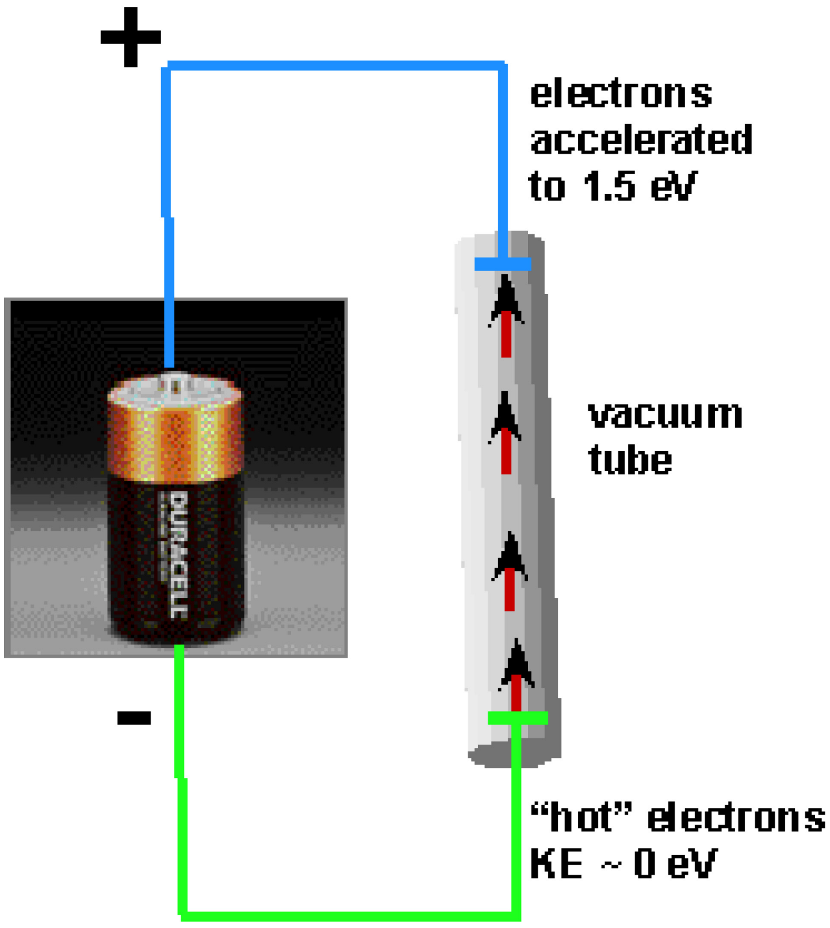
η_y, η'_y

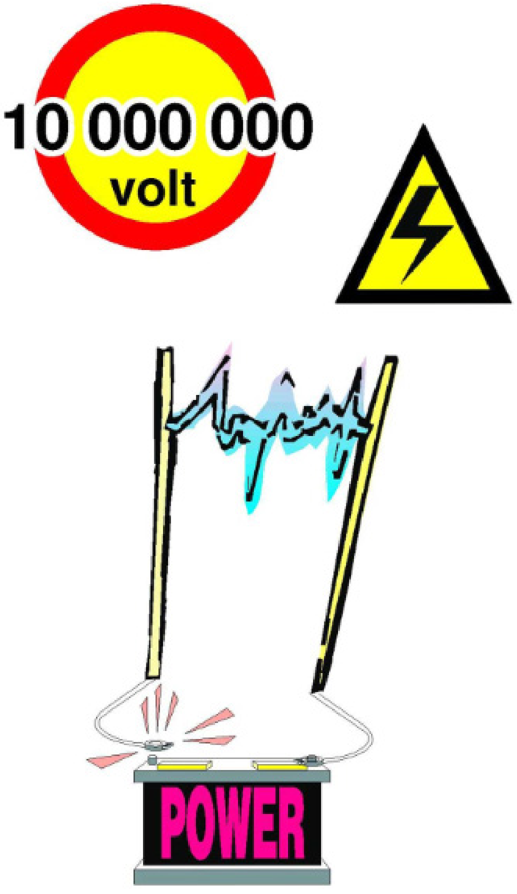
$$\delta L = \int dL = \frac{\delta p}{p_0} \int \eta_x d\theta = \frac{\delta p}{p_0} \int \frac{\eta_x}{R} ds \equiv \delta_p R_{56}$$

Conventional Acceleration

(DC acceleration, RF acceleration: Synchrotron, Linac)

DC Acceleration





[Example] Electrostatic accelerators

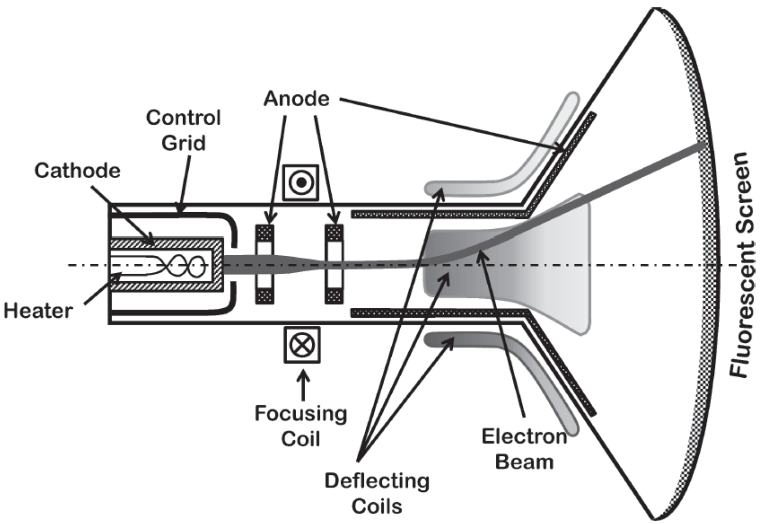


FIGURE 5.1
A cathode ray tube TV as an example of an accelerator.

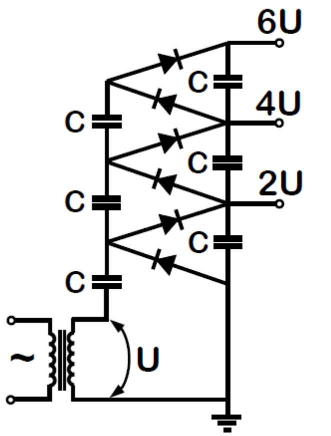


FIGURE 5.2
Cockcroft–Walton generator.

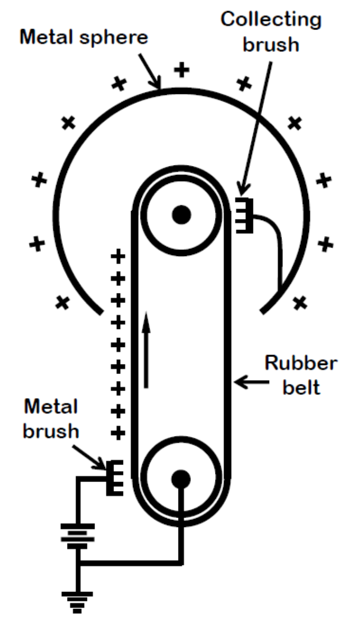


FIGURE 5.3
Van der Graaf accelerator.

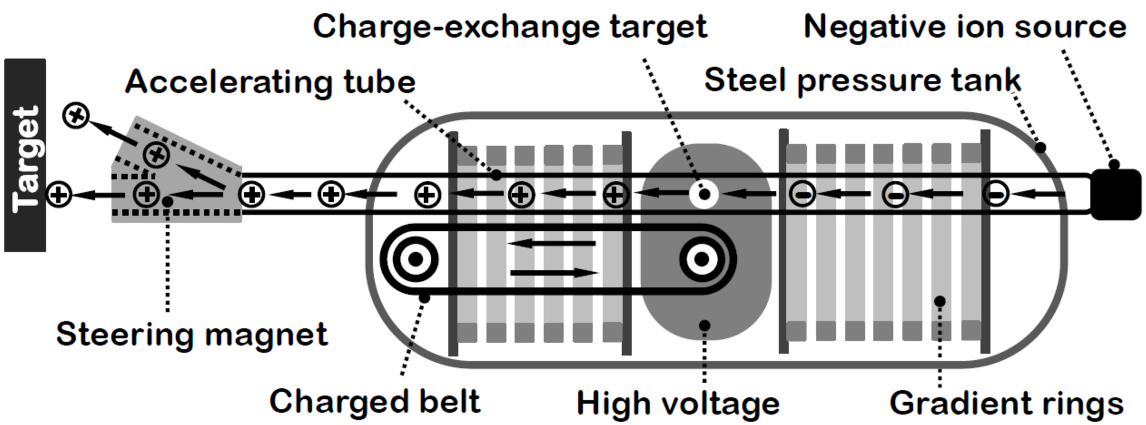
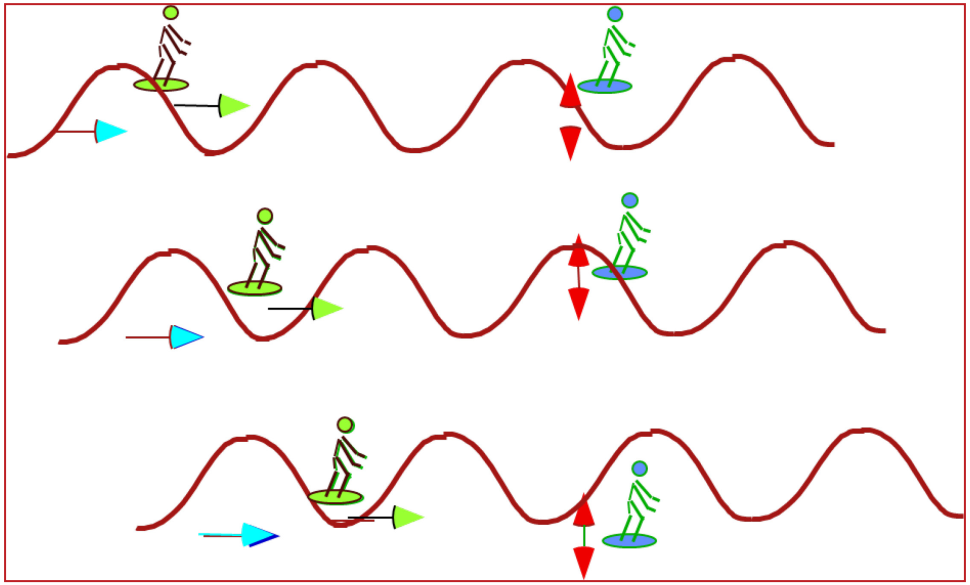


FIGURE 5.4
Tandem electrostatic accelerator.

RF Acceleration



[Example] Synchrotron and Linac

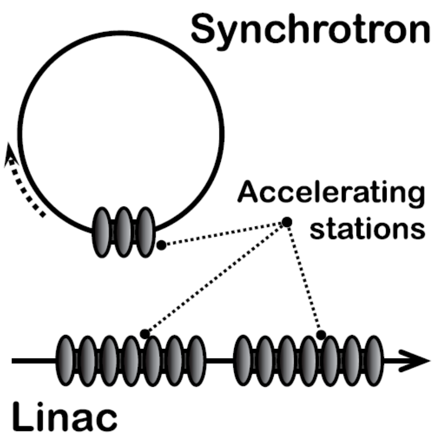


FIGURE 5.6
Synchrotron and linac.

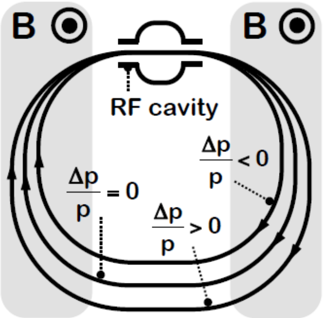
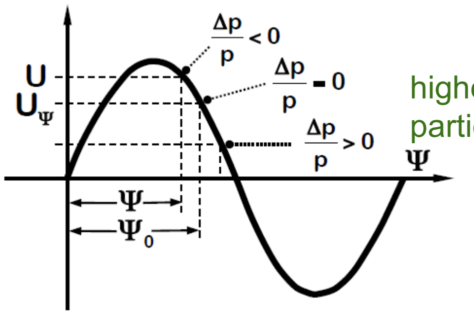


FIGURE 5.11
Synchrotron oscillations.

Lower momentum particle arrives earlier



higher momentum particle arrives later

Above transition case

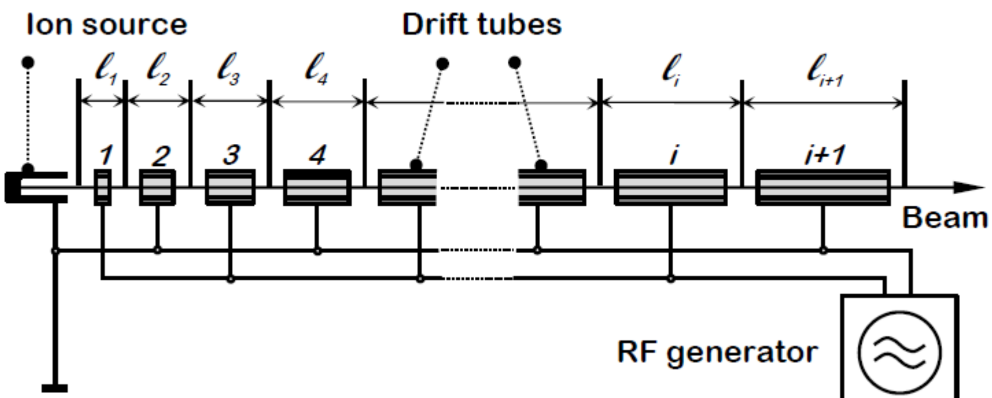


FIGURE 5.7
Wideröe linear accelerator.

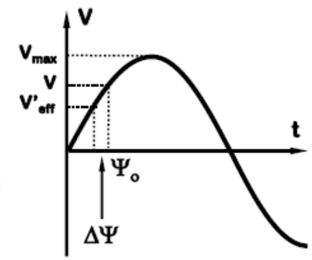
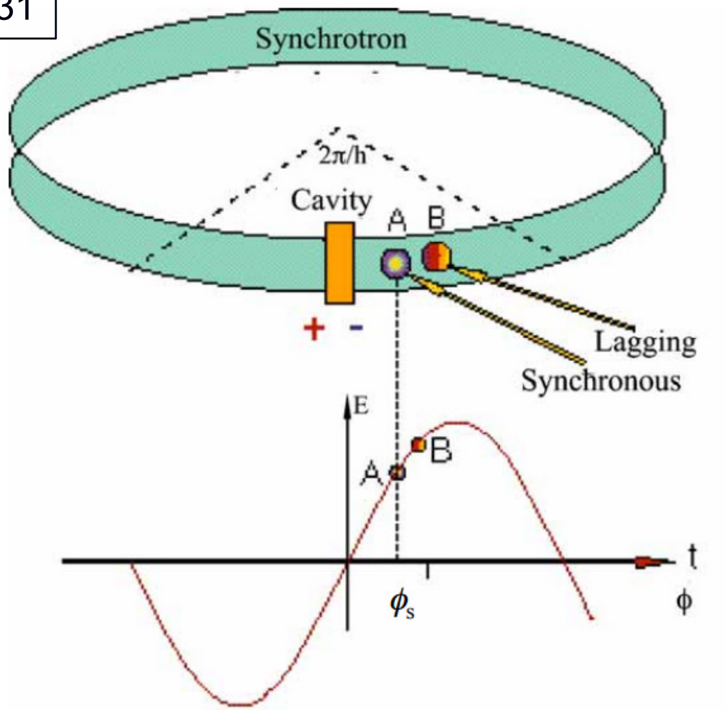


FIGURE 5.8
Voltage in Wideröe linac.

Linac (Below transition)

Synchrotron oscillation

Fig. 5.31



The synchronous particle is the particle that arrives at the RF cavity when the voltage is such that it exactly compensate the average energy loss

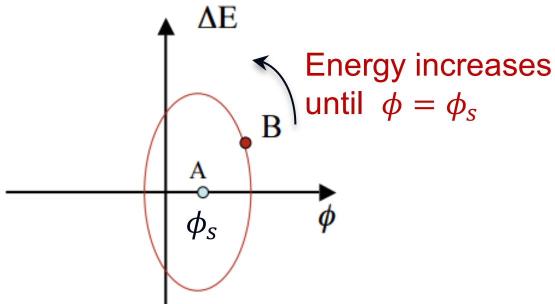
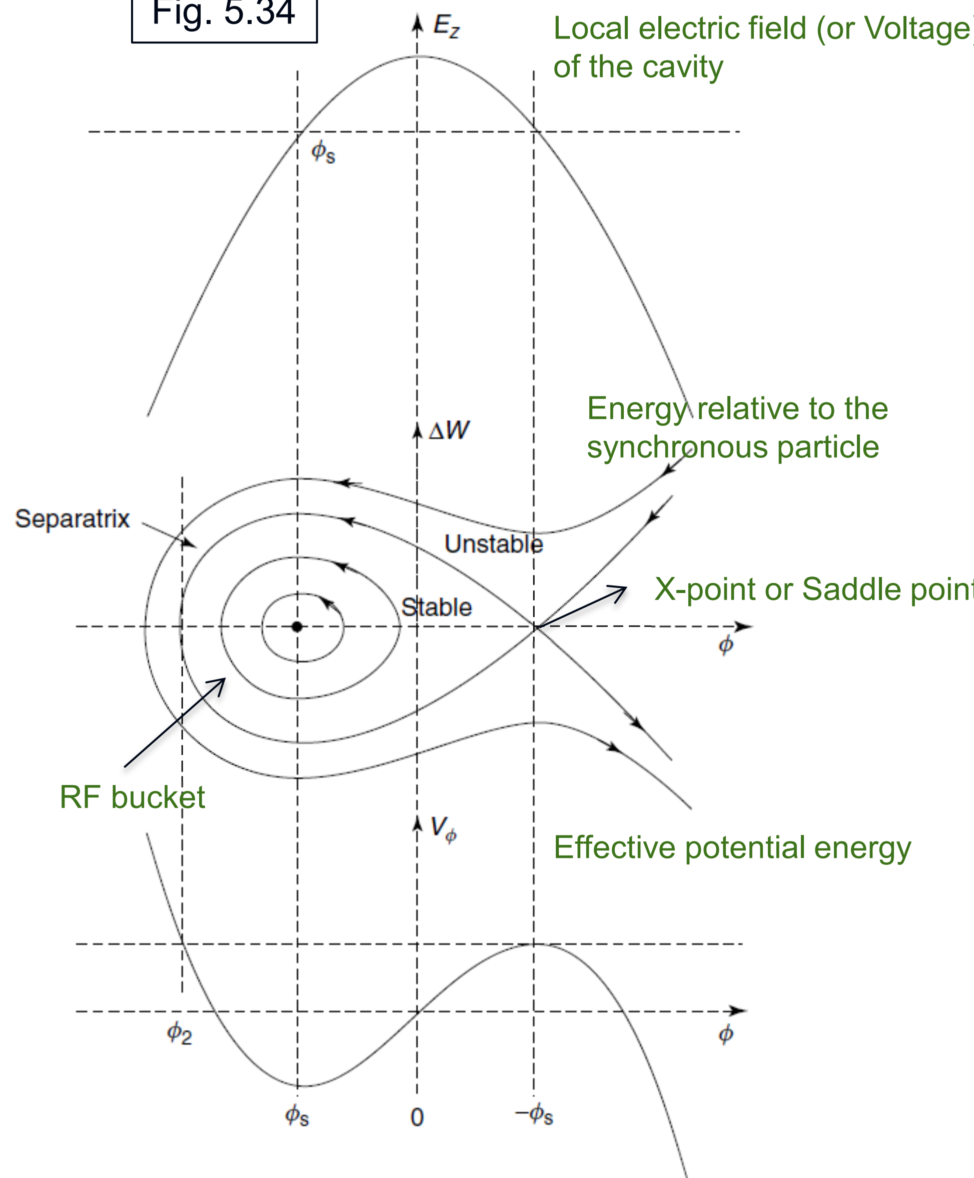


Fig. 5.33

Below transition

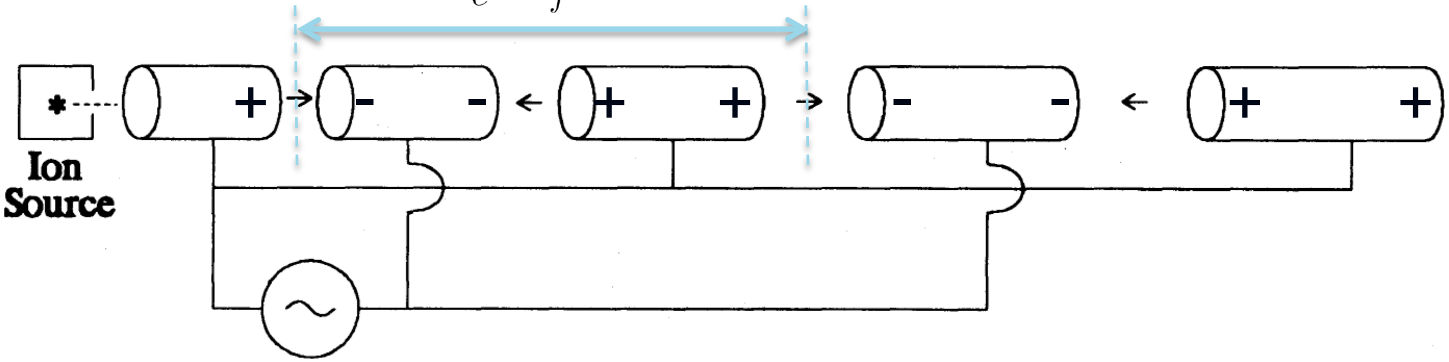
Fig. 5.34



Linac (Below transition)

Wideröe linac (not yet using a cavity)

$$2l \approx \beta\lambda = v\frac{\lambda}{c} = \frac{v}{f} = vT = \text{Distance that particle travels during 1 RF cycle}$$

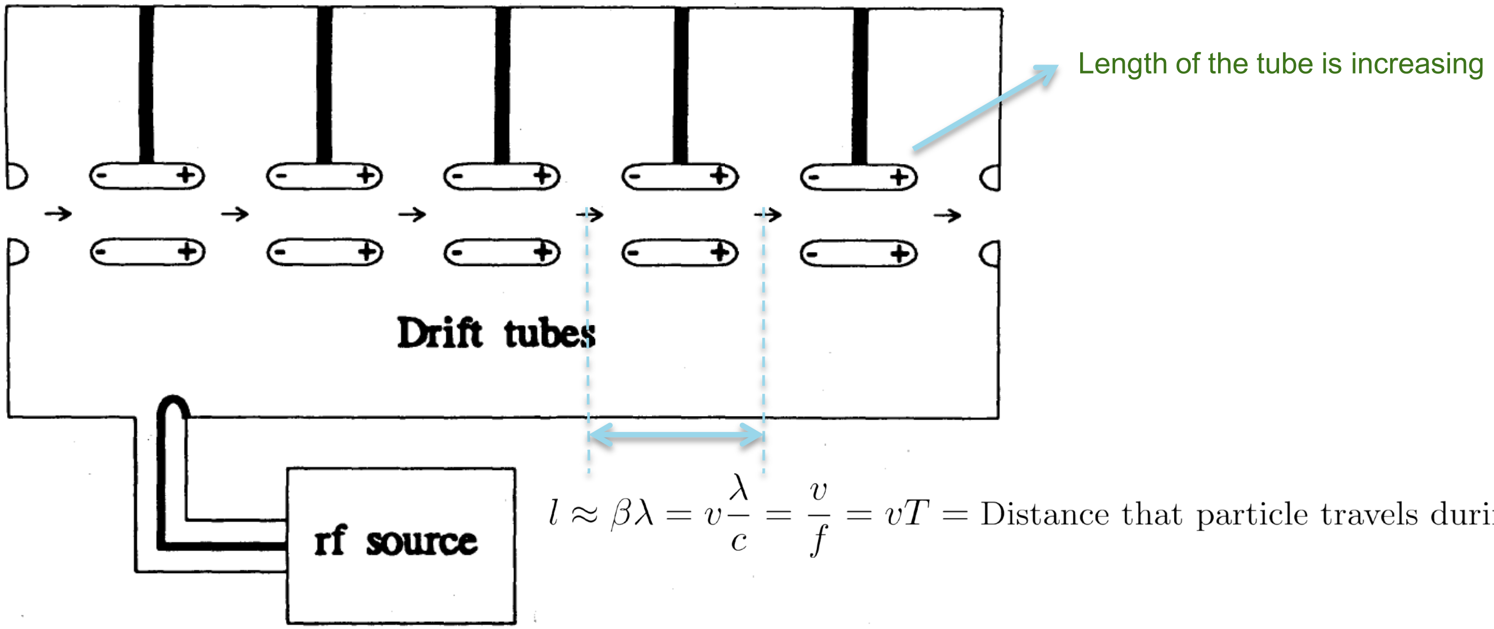


Wideröe

- Time-varying electric field is applied through transmission lines. The electrical charges are actually travelling from one tube to the next by passing through the RF generator.
- The RF phase changes by 180º (π-mode), while the particles travel from one tube to the next.

- When using low frequencies, the length of the drift tubes becomes prohibitive for high-energy particles.
- When using high frequencies, the drift tubes would act more like antennas and radiate energy instead of using it for acceleration.

Alvarez drift tube linac (inside a cavity)



Alvarez

$$l \approx \beta\lambda = v\frac{\lambda}{c} = \frac{v}{f} = vT = \text{Distance that particle travels during 1 RF cycle}$$

- The RF power is inductively coupled through a transformer consisting of a one turn primary inserted through the wall of the resonant tank containing the drift tubes.
- While the electric fields point in the "wrong direction" the particles are shielded by the drift tubes.
- The RF phases are same (0-mode), while the particles travel from one tube to the next.

Conditions for RF Acceleration

- The wave must have an electric field component along the direction of particle motion.

$$E_{||} \neq 0$$

  - This condition is not satisfied by EM waves in free space, but can be satisfied by a transverse magnetic (TM) wave propagating in a uniform waveguide.

- For a sustained energy transfer and an efficient particle acceleration, the phase velocity of the wave must be closely matched to the beam velocity.

$$v_{ph} \approx v \leq c$$

  - This condition is not satisfied for a uniform waveguide, because the phase velocity $v_{ph} > c$.
  - In periodically-loaded waveguide, reflections from the loading elements reduce the phase velocity.

- The distance between accelerating gaps (l) is proportional to particle velocity.

$$n \times l = \frac{\beta c}{f} = \beta\lambda = vT \quad (n = 1, 2, 3, \cdots)$$

  - Here, we neglect the increase in β inside the waveguide structure.

n=2: π-mode
n=1: 0-mode

Waveguides and Cavities

(Sec. 5.2 and Sec. 5.3 of UP-ALP)

Plane waves in free space

- For free space (no boundary):

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right], \quad \mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

- Phase velocity:

$$v_{ph} = \frac{\omega}{|\mathbf{k}|} = \frac{\omega}{\sqrt{k_x^2 + k_y^2 + k_z^2}} = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

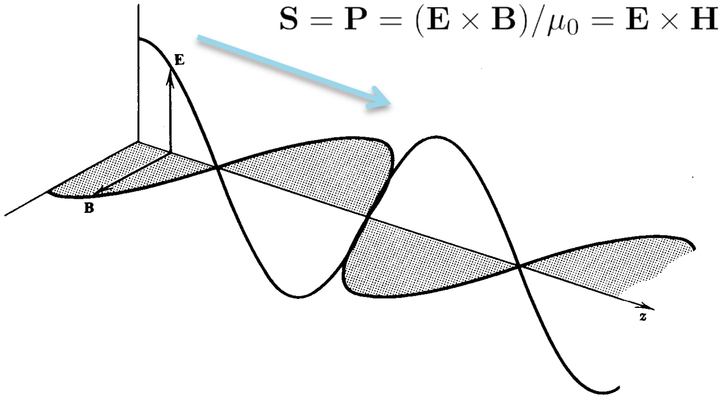
- Consequences of Maxwell equations:

$$\nabla \cdot \mathbf{E} = 0 \longrightarrow \mathbf{k} \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0 \longrightarrow \mathbf{k} \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} \longrightarrow \mathbf{k} \times \mathbf{E} = \omega \mathbf{B}, \quad \nabla \times \mathbf{B} = -i\omega\mu_0\epsilon_0 \mathbf{E} \longrightarrow \mathbf{k} \times \mathbf{B} = -\frac{\omega}{c^2}\mathbf{E}$$

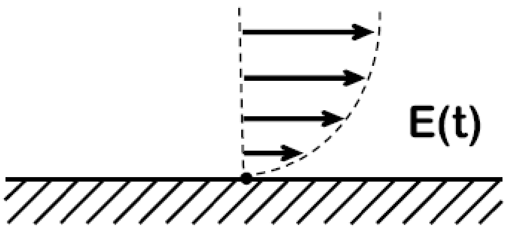
$$\boxed{\mathbf{E} \perp \mathbf{k}, \quad \mathbf{B} \perp \mathbf{k}, \quad \mathbf{E} \perp \mathbf{B}, \quad E = cB}$$

$$\mathbf{S} = \mathbf{P} = (\mathbf{E} \times \mathbf{B})/\mu_0 = \mathbf{E} \times \mathbf{H}$$

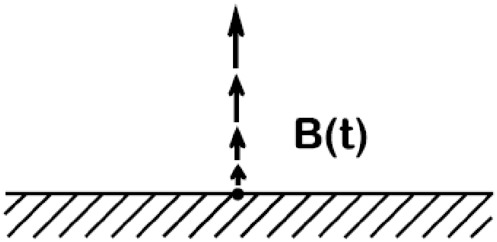


Boundary conditions at conducting surfaces

- On the surface of a perfect conductor, the tangential component of an electric field and the normal component of a magnetic field will vanish.



$$E_{||} = 0$$



$$B_{\perp} = 0$$

- A non-ideal surface has a finite conductivity:

$$\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2} = \frac{1}{\sqrt{\pi f \mu_0 \mu_r \sigma}},$$

$$\frac{dP_{\text{loss}}}{da} = \frac{1}{2} \times \underbrace{\frac{1}{\sigma \delta}}_{=R_{surf}} |\mathbf{K}_{\text{eff}}|^2$$

$$\mathbf{J} = \sigma \mathbf{E}_c, \;\; \mathbf{K}_{\text{eff}} = \int_0^\infty \mathbf{J} d\xi = \hat{\mathbf{n}} \times \mathbf{H}_{||}$$

(See Lecture 1)

Group velocity

- Interference between two continuous waves slightly different frequencies and wavenumbers:

$$\begin{aligned} E &= E_1 + E_2 \\ &= E_0 \sin\left[(k+dk)x - (\omega + d\omega)t\right] + E_0 \sin\left[(k-dk)x - (\omega - d\omega)t\right] \\ &= 2E_0 \sin[kx - \omega t] \cos[dk\ x - d\omega\ t] \\ &= 2E_0 f_1(x,t) f_2(x,t) \end{aligned}$$

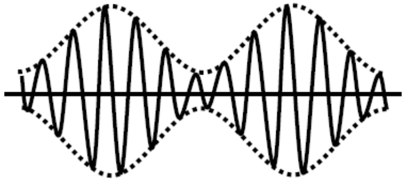


FIGURE 5.14
Two-wave interference.

- Phase velocity: by requesting the convective derivative of f_1 to be equal to zero

$$0 = \left(\frac{\partial}{\partial t} + v_p \frac{\partial}{\partial x}\right) f_1 \rightarrow v_p = -\frac{\partial f_1(x,t)/\partial t}{\partial f_1(x,t)/\partial x} = \frac{\omega}{k}$$

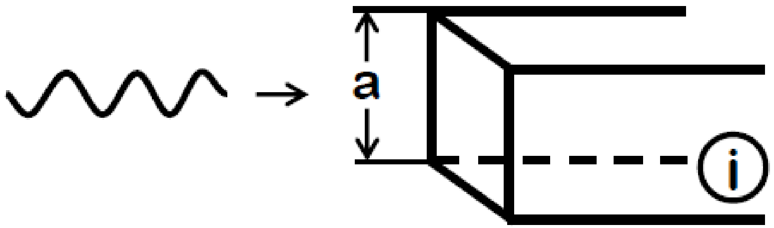
- Group velocity: by requesting the convective derivative of f_2 to be equal to zero

$$0 = \left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right) f_2 \rightarrow v_g = -\frac{\partial f_2(x,t)/\partial t}{\partial f_2(x,t)/\partial x} = \frac{d\omega}{dk}$$

Dispersion for a waveguide (Qualitative)

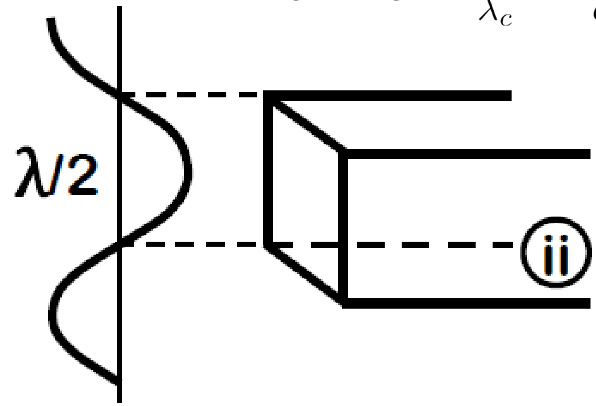
$$\lambda \ll a$$

$$\omega = ck$$



$$\lambda/2 \leq a, \quad \text{or} \quad \lambda_c = 2a$$

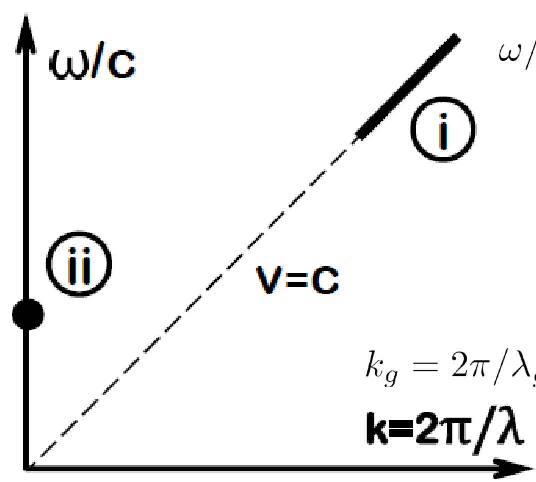
$$\omega_c = ck_c = c\frac{2\pi}{\lambda_c} = c\frac{\pi}{a}$$



If the wavelength of an EM wave in free space is much shorter than the transverse size *a* of the waveguide then the waveguide does not matter.

When half of a wavelength in free space equals the waveguide transverse size, that is the longest wavelength for which the boundary conditions at a perfectly conducting surface of the waveguide can still be satisfied.

$$\omega_c/c = k_c = \frac{2\pi}{\lambda_c} = \frac{\pi}{a}$$



$$\omega/c = k$$

$$k_g = 2\pi/\lambda_g$$

TM Mode Solution in Circular Waveguide

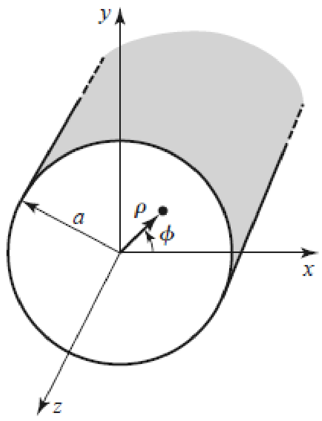
- From conducting boundary, electromagnetic wave can be transformed into TM (Magnetic field is Transverse to z) mode.
- TM fields can be found from one vector component of the magnetic vector potential (note that $\nabla \cdot \mathbf{A} \neq 0$, i.e. using Lorentz gauge) :

$$\mathbf{A} = A_z \hat{z}$$

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho/\epsilon_0, \quad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0 \mathbf{J}$$

- Helmholtz wave equation In cylindrical coordinates:

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}\right) A_z + k_0^2 A_z = 0$$



- Separation of variables with arbitrary constant C (complex in general):

$$A_z = C \times J_m(k_\rho \rho)\cos(m\phi)e^{\pm i k_g z}$$

$$-k_g^2 + k_0^2 = k_\rho^2$$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \left(k_\rho^2 - \frac{m^2}{\rho^2}\right)\right] J_m(k_\rho \rho) = 0$$

TM Mode Solution in Circular Waveguide

- Field components can be expressed by A_z alone:

$$\mathbf{B} = \nabla \times \mathbf{A} \longrightarrow B_\rho = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}, \quad B_\phi = -\frac{\partial A_z}{\partial \rho}, \quad B_z = 0$$

$$\mathbf{E} = \frac{i}{\omega \mu_0 \epsilon_0} \nabla \times \mathbf{B} \longrightarrow E_\rho = -\frac{i}{\omega \mu_0 \epsilon_0}\frac{\partial B_\phi}{\partial z}, \quad E_\phi = +\frac{i}{\omega \mu_0 \epsilon_0}\frac{\partial B_\rho}{\partial z}$$

$$E_z = \frac{i}{\omega \mu_0 \epsilon_0}[\nabla \times (\nabla \times \mathbf{A})]_z \longrightarrow E_z = \frac{i}{\omega \mu_0 \epsilon_0}\left[\frac{\partial^2 A_z}{\partial z^2} - \nabla^2 A_z\right] = \frac{i}{\omega \mu_0 \epsilon_0} k_\rho^2 A_z$$

- Boundary conditions:

$$E_\phi(\rho = a) = E_z(\rho = a) = B_\rho(\rho = a) = 0$$

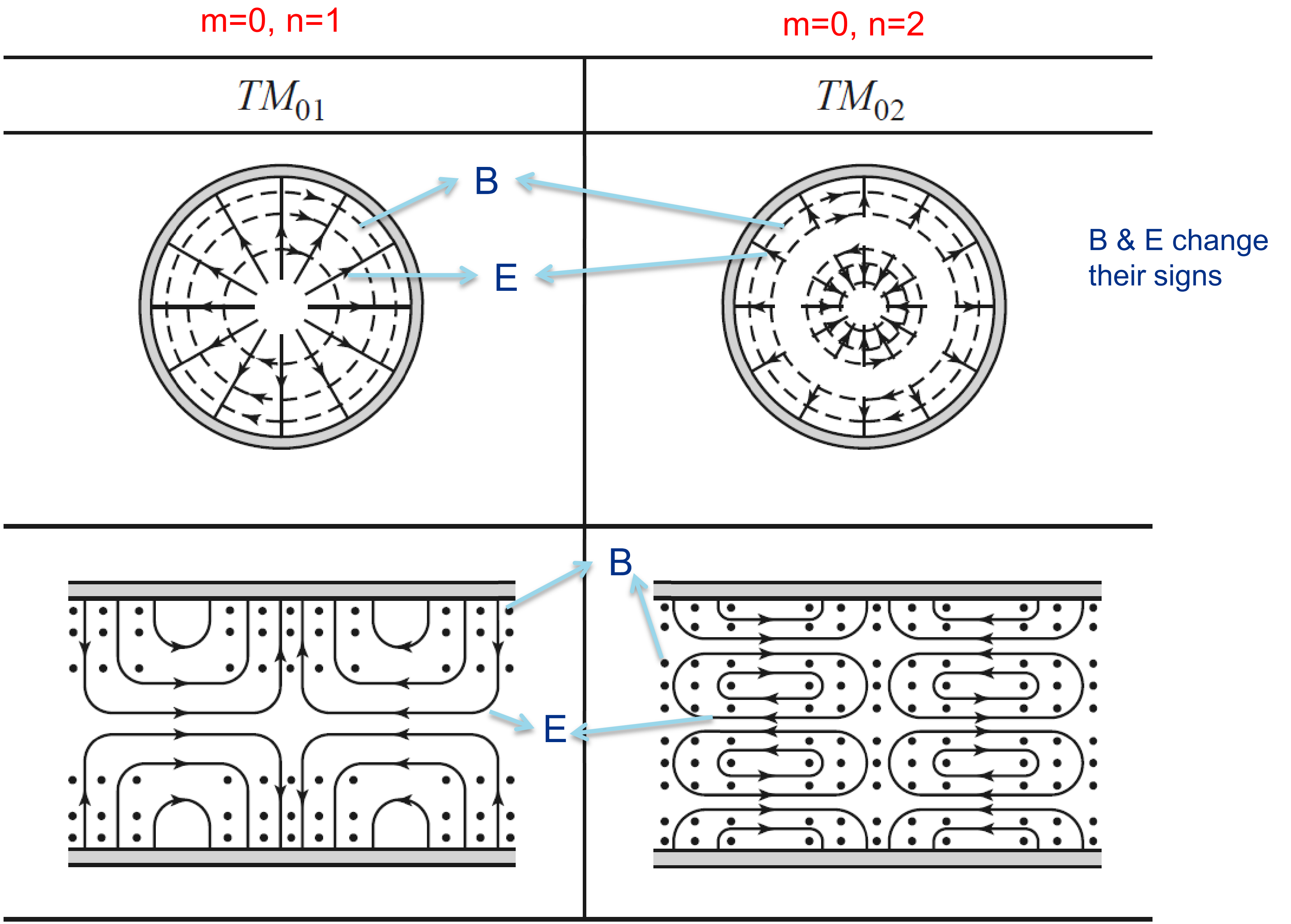
$$J_m(k_\rho a) = 0 \longrightarrow k_\rho = \frac{x_{mn}}{a} = \frac{\omega_c}{c}$$

  – x_{mn} : n-th zero of the Bessel function of order m. (e.g., $x_{01} = 2.405$)

- Dispersion relation for guide propagation constant and wavelength:

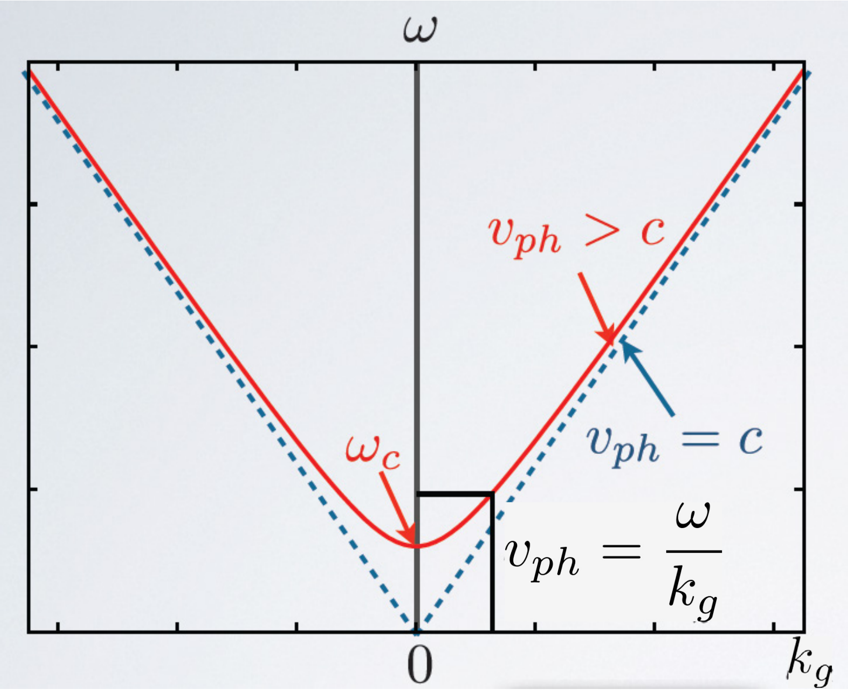
$$k_g^2 = k_0^2 - k_\rho^2 \longrightarrow \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda_g}\right)^2 + \frac{\omega_c^2}{c^2}$$

[Example]



Dispersion for a waveguide (Quantitative)

$$\omega^2 = c^2 k_g^2 + \omega_c^2$$

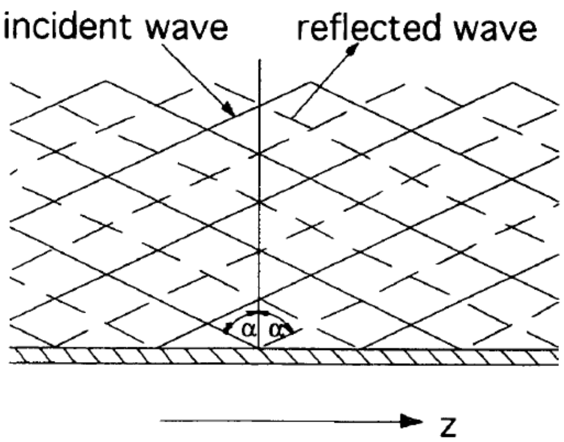


$$v_{ph} v_g = c^2$$

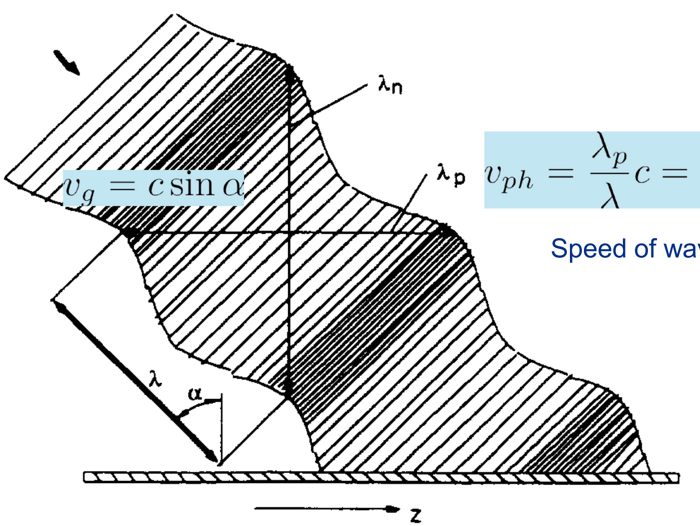
- There is a "cut-off frequency", below which a wave will not propagate. It depends on dimensions.
- At each excitation frequency is associated a phase velocity ω/k_g, , the velocity at which a certain phase travels in the waveguide.
- Energy (and information) travel at group velocity $d\omega/dk_g$, which is between 0 and c. This velocity has respect the relativity principle!
- Synchronism with RF (necessary for acceleration) is impossible because a particle would have to travel at $v = v_{ph} > c$!
- To use the waveguide to accelerate particles, we need a "trick" to slow down the wave.

Meaning of $v_{ph} > c$

- An EM wave with an oblique incidence on a conducting plane: Incident and reflected waves are combined in such a way that $E_{\parallel} = E_p = 0, \quad B_{\perp} = B_n = 0$



- Let's consider the z direction.



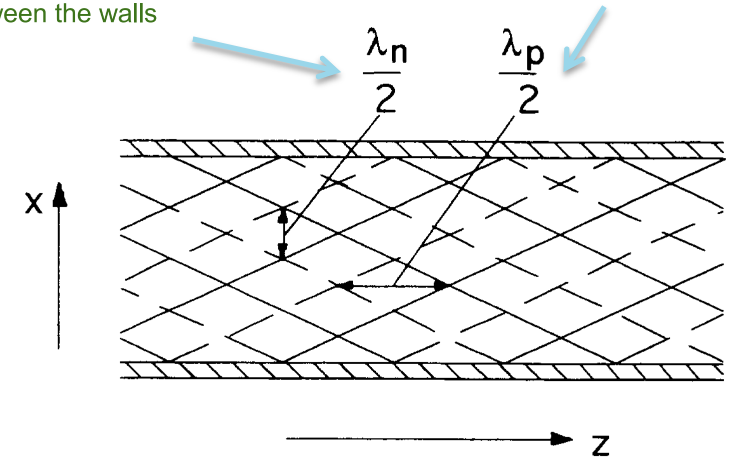
$v_g = c\sin\alpha$

$v_{ph} = \frac{\lambda_p}{\lambda}c = \frac{c}{\sin\alpha} > c$

Speed of wave-front

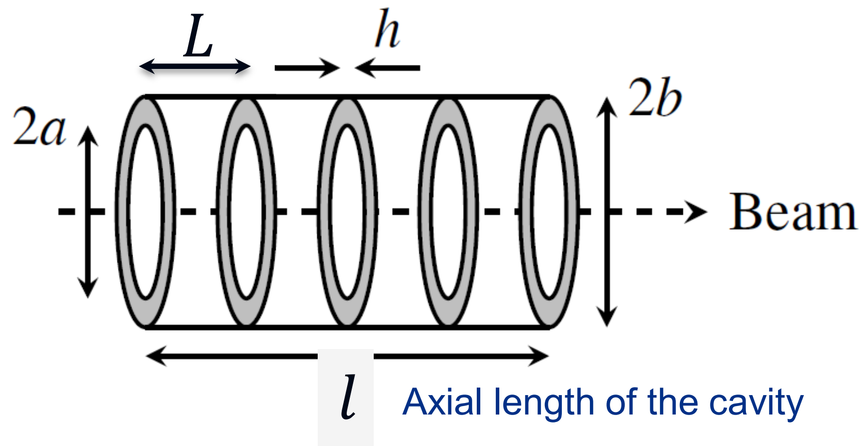
Standing wave structure in the x direction: Integer number of half-wave length between the walls

Travelling wave structure in the z direction:



Iris(Disk)-Loaded Waveguide

- In order to slow down the waves in simple waveguide, we introduce some periodic obstacles. Iris acts as a scatter, resulting in a transmitted as well as a reflected wave.



Axial length of the cavity

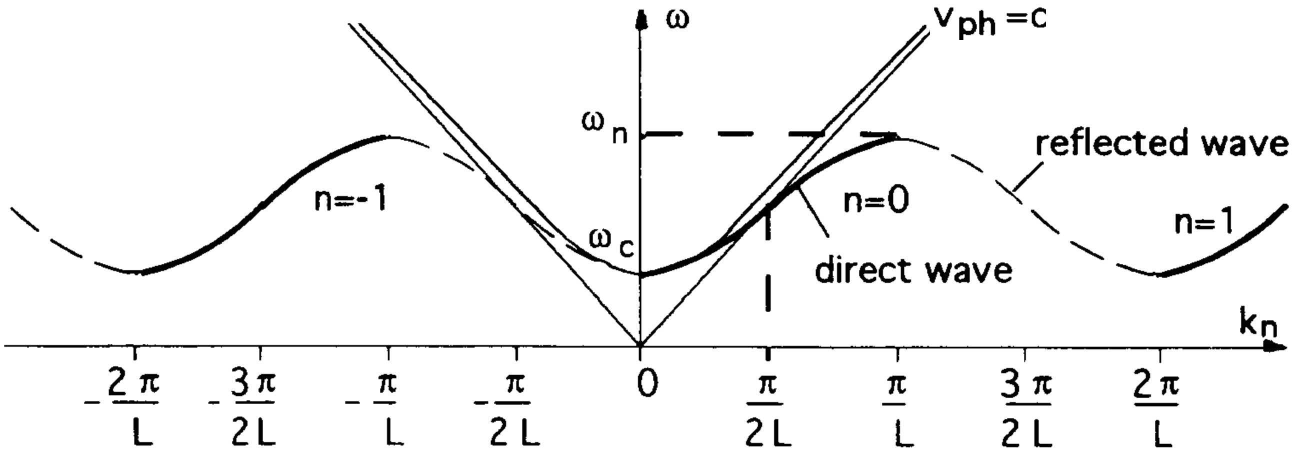
- The complicated boundary conditions cannot be satisfied by a single mode, but by a whole spectrum of space harmonics.

- From Chap.3.11 of Wangler's textbook [RF Linear Accelerators]:

$$\omega = \frac{2.405c}{b}\sqrt{1+\kappa[1-\cos(k_n l)e^{-\alpha h}]}$$

$$\kappa = \frac{4a^2}{3\pi J_1^2(2.405)b^2 l} \ll 1, \quad \alpha \approx \frac{2.405}{a}, \quad k_n = k_z + \frac{2\pi n}{L}$$

Brillouin Diagram



- For a given mode, there is a limited passband of possible frequencies; at both ends of the passband, the group velocity is 0.
- For a given frequency, there is an infinite series of space harmonics ($-\infty < n < +\infty$). All space harmonics have the same group velocity, but different v_{ph}.
- The directed (reflected) wave are characterized by $v_g > 0$ ($v_g < 0$), i.e., the EM energy flows in the $+z$ ($-z$) direction.
- At the end of the waveguide, the EM energy can either be dissipated into a matched load (travelling-wave structure) or be reflected back and forth by shortening end walls (standing-wave structure) → Energy can also be transferred to a particle beam from an standing wave in an RF cavity (next topic).

TM Mode of Pillbox Cavity

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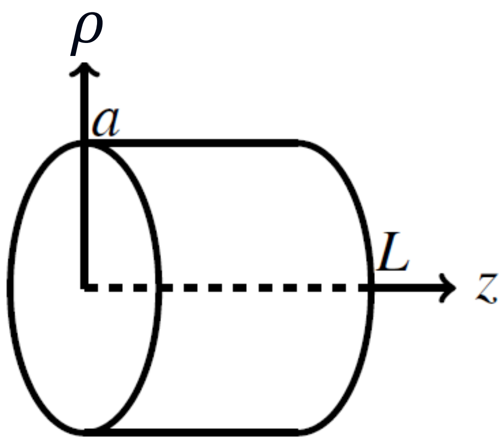
- We simply superpose two waves in a circular waveguide, one propagating in the positive z direction and the other propagating in the negative z.

$$A_z = C \times J_m(k_\rho \rho) \cos(m\phi) \left(e^{+ik_g z} + e^{-ik_g z}\right) = 2C \times J_m(k_\rho \rho) \cos(m\phi) \cos(k_g z)$$



- Additional boundary conditions at $z = 0$ and $z = L$:

$$E_\rho(z=0) = E_\phi(z=0) = E_\rho(z=L) = E_\phi(z=L) = 0$$

$$E_\rho, E_\phi \propto \sin(k_g z) \;\longrightarrow\; k_g L = p\pi \;\; (p = 0, 1, 2, \cdots)$$

- Dispersion relation: discrete resonance frequency (it was continuous for WG)

$$\frac{\omega^2}{c^2} = \left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{p\pi}{L}\right)^2$$

[Example] TM_{010} Mode

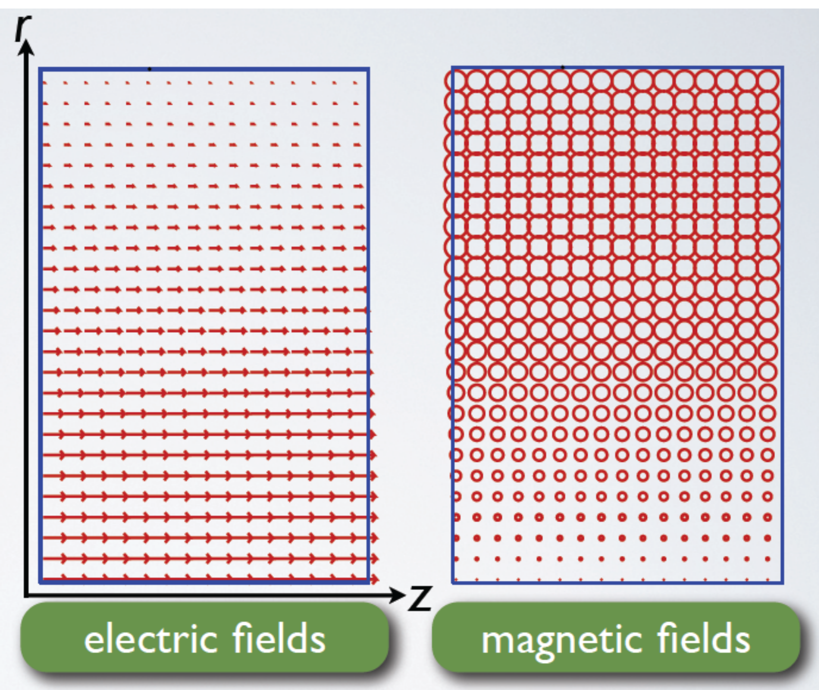
- Simplest and lowest frequency mode: $TM_{mnp} = TM_{010}$

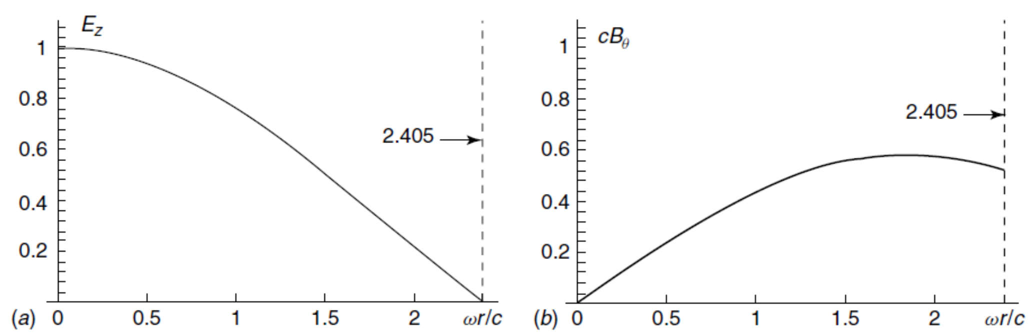
$$k_\rho = \frac{2.405}{a}, \quad \omega = \omega_{010} = \frac{2.405c}{a}$$

- Explicit expression for fields:

$$E_z = E_0 J_0(k_\rho \rho) e^{-i\omega t}, \quad B_\phi = -i\frac{E_0}{c} J_1(k_\rho \rho) e^{-i\omega t}$$

Phase difference

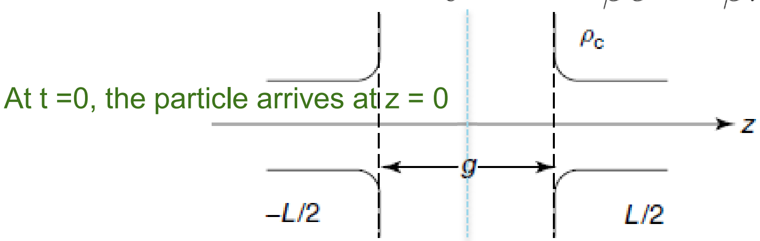


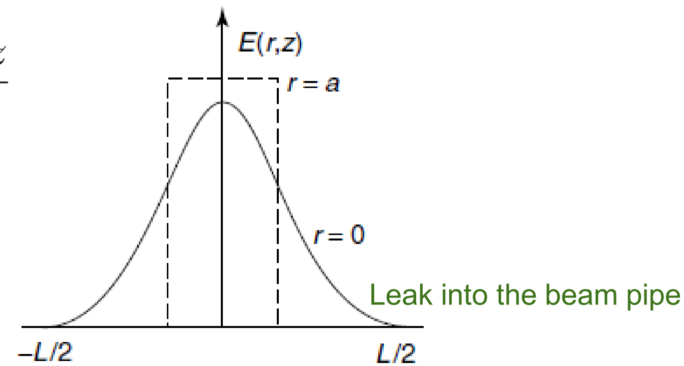


Cavity Parameters: Transit Time Factor

- We suppose that the field is symmetric about $z = 0$, and confined within an axial distance L containing the gap, in which velocity change is small.

$$\omega t \approx \omega \frac{z}{v} = 2\pi f \frac{z}{\beta c} = \frac{2\pi z}{\beta \lambda}$$





$$\Delta W = q \int_{-L/2}^{L/2} E(0,z)\cos(\omega t + \varphi_s)dz = qV_0 T \cos\varphi_s$$

where

$$V_0 = \int_{-L/2}^{L/2} E(0,z)dz = E_0 L, \quad T = \frac{\int_{-L/2}^{L/2} E(0,z)\cos(\omega t)dz}{V_0} \approx \frac{\int_{-L/2}^{L/2} E(0,z)\cos(2\pi z/\beta\lambda)dz}{V_0}$$

- Accelerating voltage and gradient: Effect of transit time factor (T) is included.

$$V_{acc} = V_0 T, \quad E_0 T\ [\mathrm{MV/m}] = \frac{V_{acc}}{L}$$

Cavity Parameters: Transit Time Factor

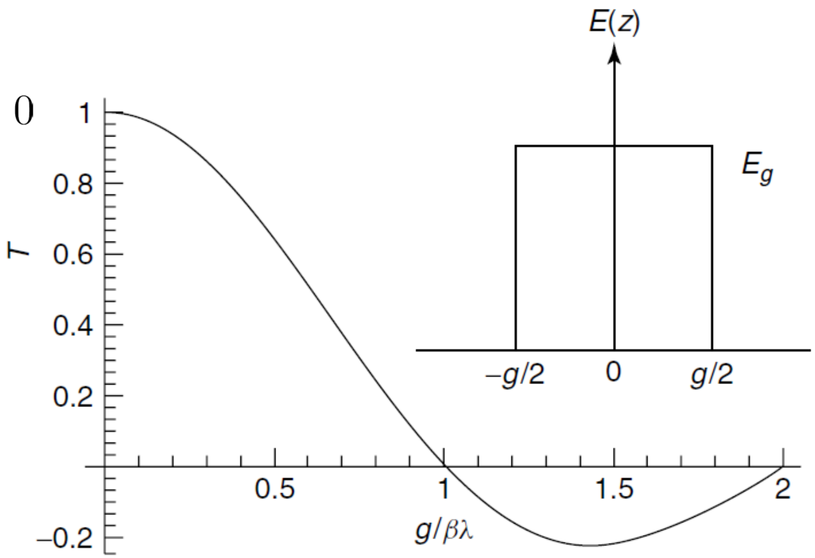
- Physical meaning: ratio of the energy gained in the time-varying RF field to that in a DC field of voltage $V_0 \cos(\varphi_s)$.
- Thus, T is a measure of the reduction in the energy gain caused by the sinusoidal time variation of the field in the gap.

Ex] A simple TM_{010} pillbox cavity of length g:

$$E(0,z) = E_g = const., \qquad T = \frac{\sin(\pi g/\beta\lambda)}{\pi g/\beta\lambda}$$

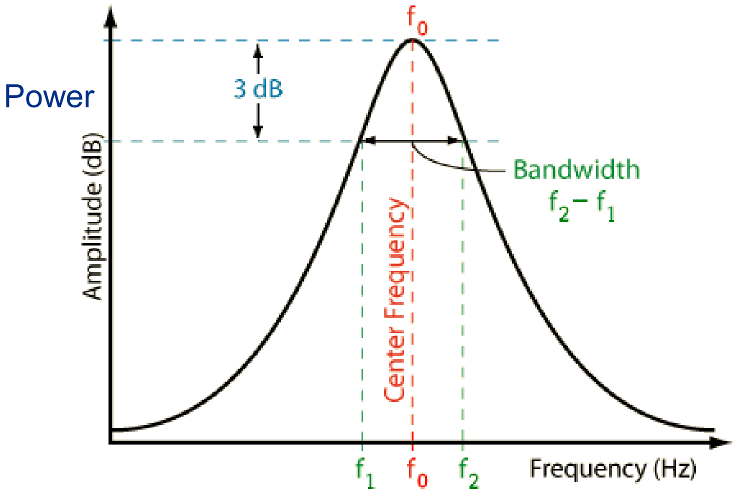
If $g/\beta\lambda \longrightarrow 0$

$$\begin{aligned} g &\rightarrow 0 \quad \text{very short gap} \\ \beta\lambda &= v\lambda/c \\ &= vT_{rf} \\ &\rightarrow \infty \quad \text{very fast transit} \end{aligned}$$



Cavity Parameters: Quality Factor

- The quality factor Q describes the bandwidth of a resonator and is defined as the ratio of the reactive power (stored energy) to the real power that is dissipated in the cavity walls.



$$Q = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f} = \frac{\omega_0 U}{P_d}$$

Ex] For SC cavities, $Q \approx 10^{10} \sim 10^{11}$. Why so high ?

- Filling/Decay time of a cavity: Narrow freq. response → Long time response

$$P_d = -\frac{dU}{dt} = \frac{\omega_0 U}{Q} \quad \longrightarrow U(t) = U_0 e^{-2t/\tau}, \quad \tau = \frac{2Q}{\omega_0}$$

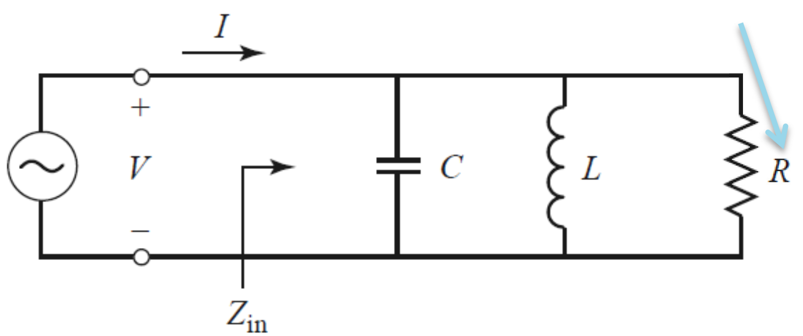
- If the cavity is connected with a power coupler, some power will leak out though the coupler and be dissipated through the external load/waveguide.

$$Q_{ext} = \frac{\omega_0 U}{P_{ext}}, \quad Q_{loaded} = \frac{\omega_0 U}{P_{ext} + P_{cav}}$$

Resonant Circuit

- A parallel resonant circuit driven by a current generator is the simplest model for describing a single mode of an accelerating cavity (damped driven oscillator).

$$R_s^{circuit} = \frac{V_0^2}{2P_d}$$



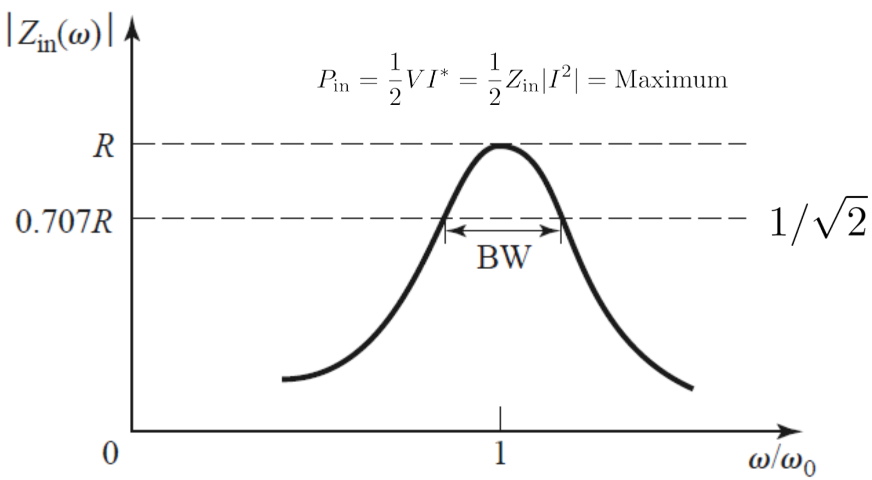
(a)

$$I(t) = I_0 e^{j\omega t}, \quad V(t) = V_0 e^{j(\omega t + \phi)}$$

Real amplitude

$$I(t) = C\frac{dV}{dt} + \frac{1}{L}\int V\,dt + \frac{V}{R}$$

$$V(t) = \underbrace{\left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)^{-1}}_{=Z_{\rm in}} I(t)$$



(b)

- Resonance frequency:

$$\omega_0 = 1/\sqrt{LC}$$

- Stored energy at resonance ($U_m = U_e$):

$$U = CV_0^2/2 = L|I_L|^2/2$$

- Dissipated power:

$$P_d = V_0^2/2R$$

- Quality factor:

$$Q = \omega_0 U/P_d = \omega_0 RC = R/\omega_0 L$$

Cavity Parameters: Shunt Impedance

- Shunt impedance: A figure of merit that measures the effectiveness of producing an axial voltage V_0 for a given power dissipated P_d.

Don't be confused with surface resistance $P_d \propto R_{surf}$

$$R_s = \frac{V_0^2}{P_d}$$

- Including the transit time factor, we define effective shunt impedance:

$$R_s^{eff} = \frac{(V_0 T)^2}{P_d}$$

- Be careful ! Accelerator community uses different definition of the shunt impedance.

$$R_s^{circuit} = \frac{V_0^2}{2P_d}$$

- R-over-Q: the ratio of R to Q (quality factor), which measures the efficiency of acceleration per unit stored energy U at a given frequency.

$$\left[\frac{R}{Q}\right] = \frac{(V_0 T)^2}{\omega U}$$

  – A single geometric quantity given in Ohms.

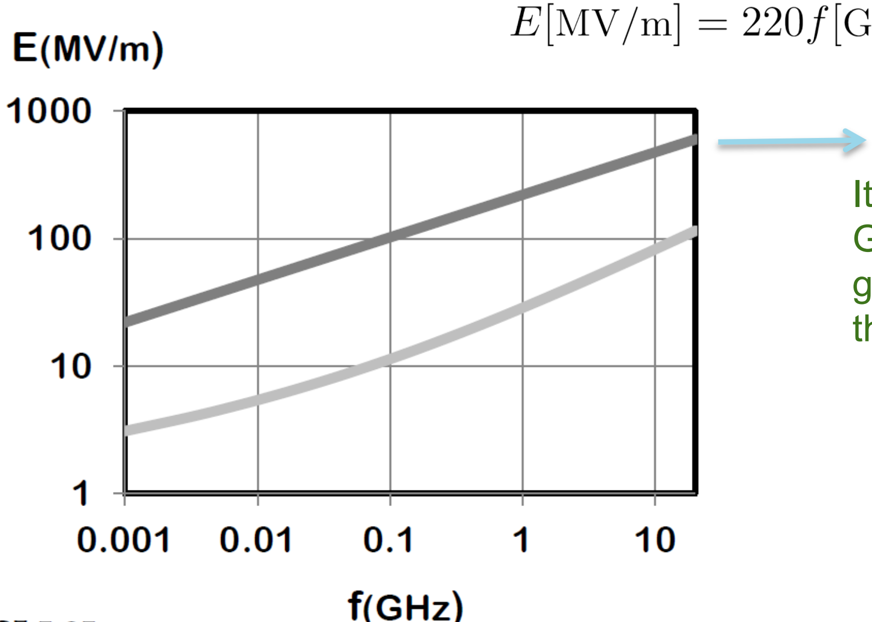
Cavity Parameters: Maximum Achievable Gradient

- Empirically derived around 1950, the Kilpatrick limit expresses the relation between the accelerating frequency and maximum achievable accelerating field of any normal conducting cavity:

$$f[\mathrm{MHZ}] = 1.64 E_k[\mathrm{MV/m}]^2 e^{-8.5/E_k[\mathrm{MV/m}]}$$

- After improving surface quality and cleanness to avoid RF breakdown, a considerable increase of achievable accelerating gradients has been made. In particular, Wang and Loew's empirical formula, devised in 1997, suggests the following behaviors:

$$E[\mathrm{MV/m}] = 220 f[\mathrm{GHz}]^{1/3}$$



It was observed that at higher (multi-tens of GHz) frequency regimes, the maximum gradients appear to be rather independent of the frequency.

FIGURE 5.25
Breakdown Kilpatrick limit (lower curve) and Wang–Loew limit (upper curve).

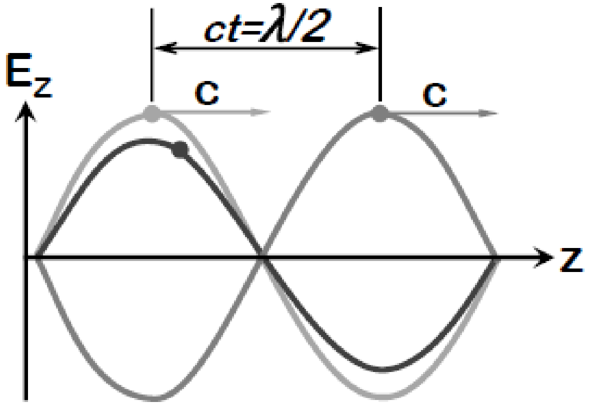
Longitudinal Dynamics: Qualitative Picture

(Sec. 5.5 of UP-ALP)

Acceleration in RF Structure

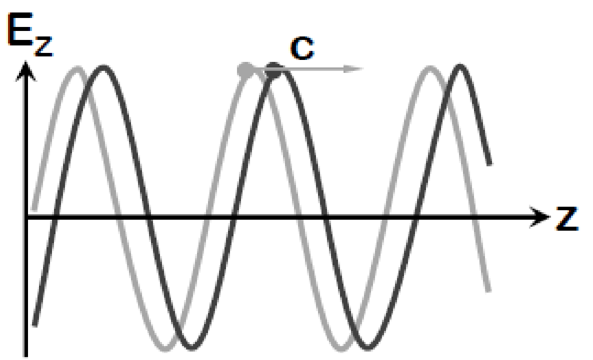
- Standing wave: The particle bunch in a standing wave observes the electric field with a varying function of time as

$$E_z = E_0 \cos(\omega t + \varphi_s) \sin(kz)$$



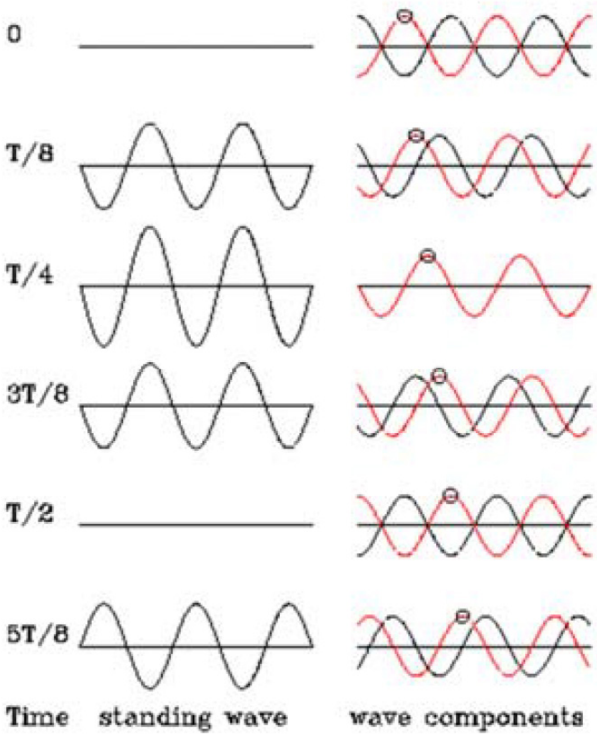
- Travelling wave: The particles in an appropriately synchronized travelling wave experience a constant electric field

$$E_z = E_0 \cos(\omega t - kz) = E_0 \cos(\varphi_s)$$



Comments on the Standing Wave Structure

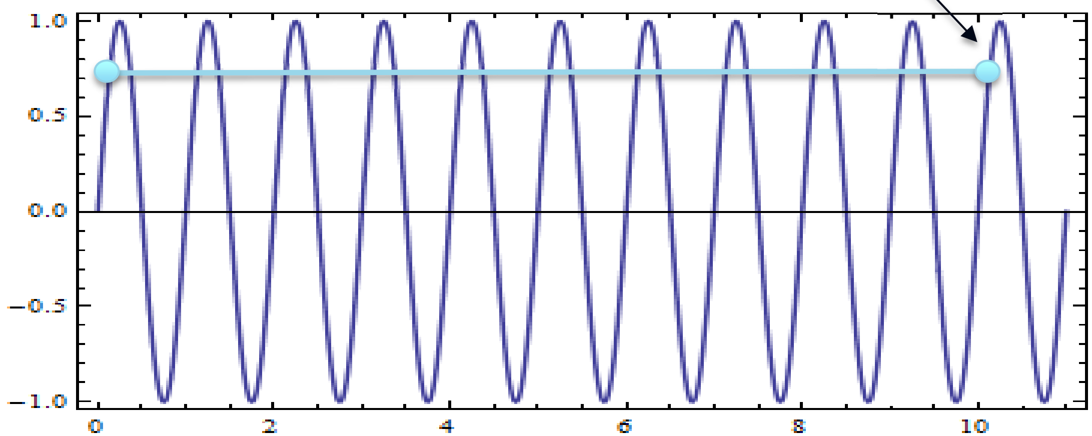
- These standing wave modes are generated by the sum of 2 traveling waves in opposite directions.
- Since only the forward wave can accelerate the beam, the shunt impedance (effectiveness of producing axial voltage for a given power dissipated) is ½ of that of the travelling wave structure.
- The standing wave could accelerate oppositely charged beams traveling in opposite directions.



Comments on the Travelling Wave Structure

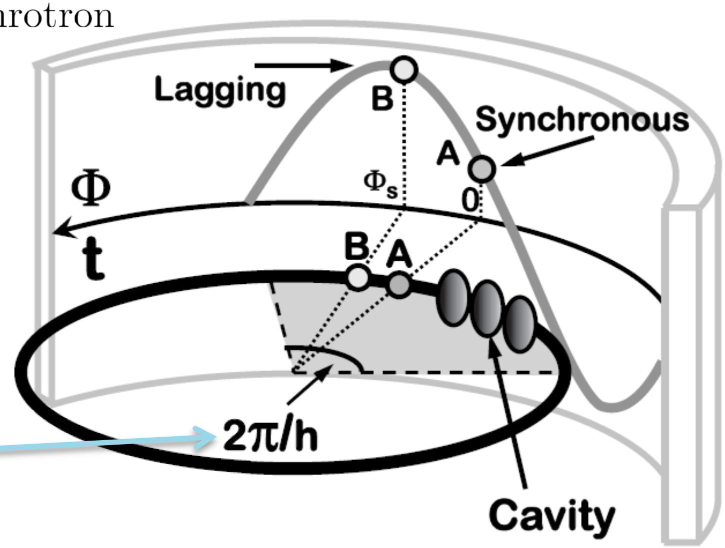
- If you consider the particle evolution in the RF cavity turn by turn, the equation of the motion is similar to the case with the travelling wave.

Bunch arrives at the cavity after 10 RF cycles



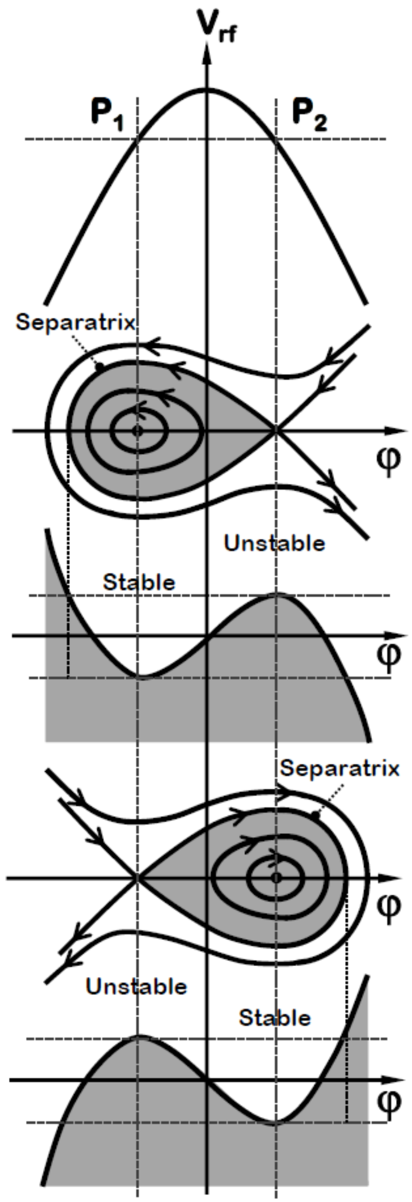
h = harmonic number ~ number of bunches in the synchrotron

$$T_{rev} = hT_{rf} \quad \text{or} \quad \omega_{rf} = h\omega_{rev}$$



Angle corresponds to 1 RF cycle

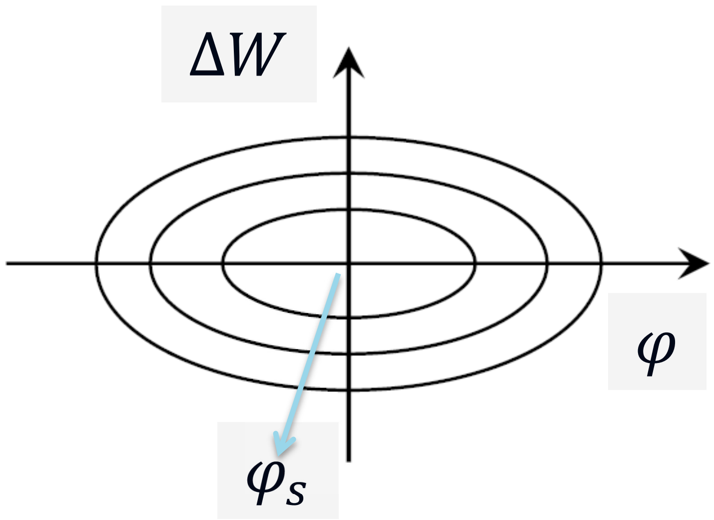
Longitudinal Dynamics in a Synchrotron



Below transition

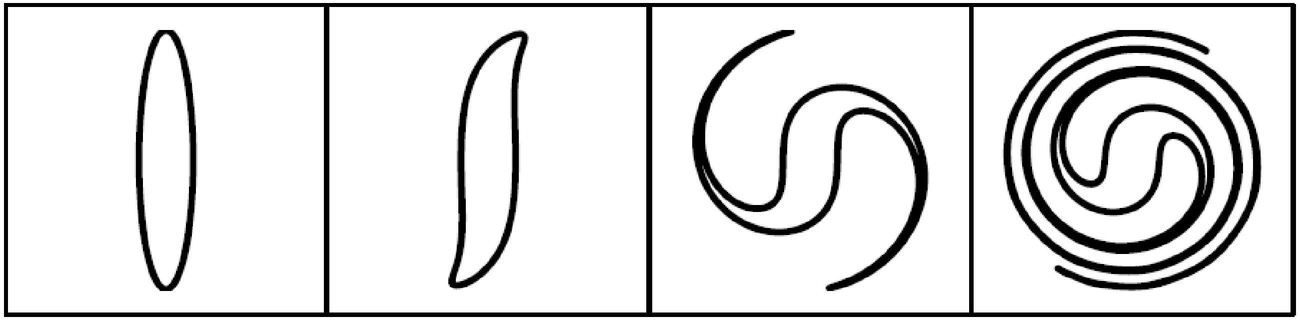
Above transition

SHO-like oscillation near the synchronous phase

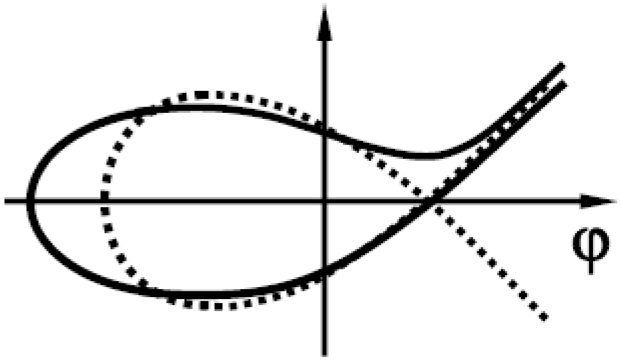


Longitudinal Dynamics in a Synchrotron

- Nonlinearity → Filamentation of the phase space → Longitudinal emittance growth



- Fast acceleration VS adiabatic acceleration: Golf-club like RF bucket



Synchrotron Tune and Betatron Tune

- Tune: Number of oscillations per one period the machine (one revolution for the circular machine). It is denoted either by Q or ν.

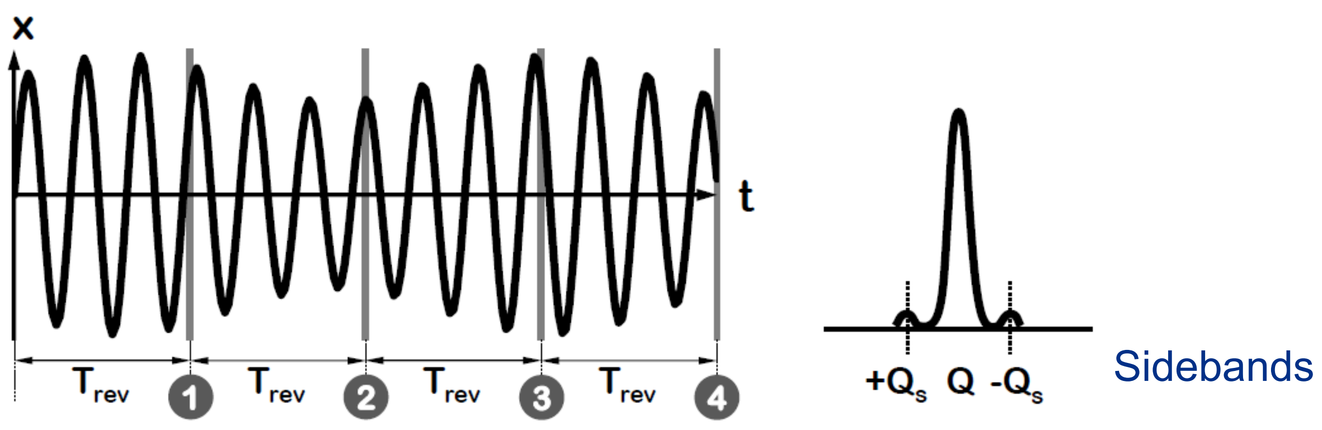
- Synchrotron tune:

$$Q_s = \frac{T_{rev}}{2\pi/\omega_s} \ll 1$$

- Betatron tune:

Phase advance

$$Q = \frac{T_{rev}}{2\pi/\omega_\beta} = \frac{\mu}{2\pi} \gg 1$$



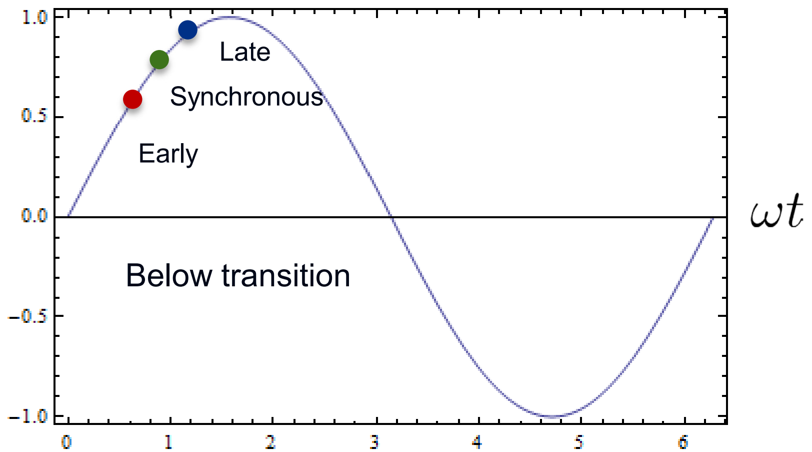
$$x \sim \sin(\omega_\beta t)\,[1+\Delta\sin(\omega_s t)] \sim \sin(2\pi Q t/T_{rev})\,[1+\Delta\sin(2\pi Q_s t/T_{rev})]$$

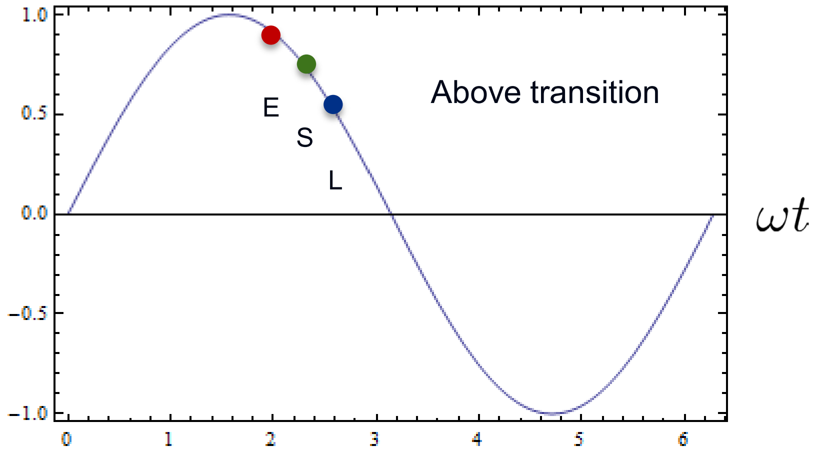
Longitudinal Dynamics: Hamiltonian Approach

(Sec. 4.2 of FOBP)

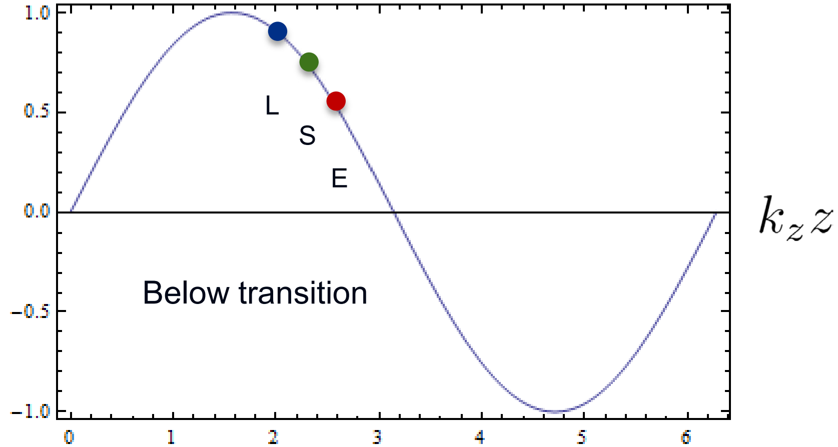
Phase Convention

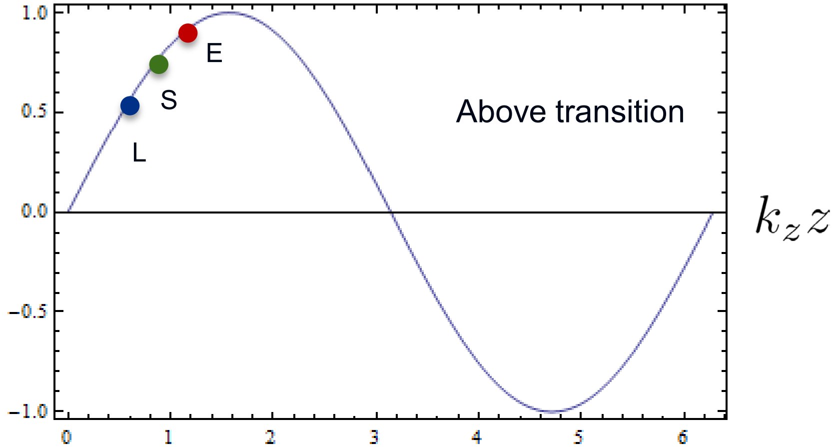
- Using arrival time:





- Using axial distance:





Acceleration in Travelling Wave

- Longitudinal electric field associated with a single travelling wave can be derived from a vector potential with only a longitudinal component:

$$A_z(z - v_\phi t) = -\frac{E_0}{k_z v_\phi}\cos[k_z(z - v_\phi t)]$$

$$v_\phi = v_{ph} = \frac{\omega}{k_z}$$

$$E_z(z - v_\phi t) = -\frac{\partial A_z}{\partial t} = E_0 \sin[k_z(z - v_\phi t)]$$

- Only considering longitudinal motion, Hamiltonian with the vector potential can be written as

$$H = \sqrt{\left(p_{z,c} + \frac{qE_0}{k_z v_\phi}\cos[k_z(z - v_\phi t)]\right)^2 c^2 + (m_0c^2)^2} \qquad \text{Eq. (4.8)}$$

$$p_{z,c} + \frac{qE_0}{k_z v_\phi}\cos[k_z(z - v_\phi t)] = p_z(\text{mechanical})$$

- Equations of motion:

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_{z,c}} = \frac{p_z c^2}{\sqrt{p_z^2c^2 + (m_0c^2)^2}} = v_z$$

$$\sqrt{p_z^2c^2 + (m_0c^2)^2} = \gamma m_0 c^2$$

$$\frac{dp_{z,c}}{dt} = -\frac{\partial H}{\partial z} = \frac{p_z c^2 (qE_0/v_\phi)\sin[k_z(z - v_\phi t)]}{\sqrt{p_z^2c^2 + (m_0c^2)^2}} = \frac{qE_0 v_z}{v_\phi}\sin[k_z(z - v_\phi t)]$$

→ This is for the canonical momentum

Acceleration in Travelling Wave

- The equation of motion for the mechanical momentum can be recovered using

$$p_{z,c} = p_z + qA_z = p_z - \frac{qE_0}{k_z v_\phi}\cos[k_z(z - v_\phi t)]$$

$$\frac{dp_z}{dt} = \frac{dp_{z,c}}{dt} - q\frac{dA_z}{dt} = \frac{dp_{z,c}}{dt} - q\underbrace{\left[\frac{\partial A_z}{\partial t} + v_z\frac{\partial A_z}{\partial z}\right]} = qE_0\sin[k_z(z - v_\phi t)]$$

Total time derivative at the particle position
= sum of the partial and the convective derivatives

- The main problem of the Hamiltonian given in Eq. (4.8) is that it is not a constant of motion, as its partial time derivative does not vanish.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \neq 0$$

  - In order to make phase plane plots of the longitudinal motion, we must convert the form of the Hamiltonian to one in which it is constant in time.
  - This is done by use of a canonical transformation. (See slides for Lecture 1)

[Review] Canonical Transformation

- The variation of the action integral between two fixed endpoints:

$$\delta \int_{t_1}^{t_2} L\ dt = \delta \int_{t_1}^{t_2} [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]\ dt = 0$$

- We would like to transform from the old coordinate system $(\mathbf{q},\mathbf{p})$ to a new system $(\mathbf{Q},\mathbf{P})$ with a new Hamiltonian $K(\mathbf{Q},\mathbf{P},t)$:

$$\delta \int_{t_1}^{t_2} [\mathbf{P}\cdot\dot{\mathbf{Q}} - K(\mathbf{Q},\mathbf{P},t)]\ dt = 0$$

- One way for both vibrational integral equalities to be satisfied is to have

$$\lambda[\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)] = \mathbf{P}\cdot\dot{\mathbf{Q}} - K(\mathbf{Q},\mathbf{P},t) + \frac{dF}{dt}$$

- If $\lambda \neq 1$, it is extended canonical transformation. If $\lambda \neq 1$ and $\frac{dF}{dt} = 0$, it is scale transformation. These transformations do not preserve phase space volume

[Review] Generating Function

- The function F is in general a function of both the old and new variables as well as the time. We will restrict ourselves to functions that contain half of the old variables and half the new; these are useful for determining the explicit form of the transformation.

Case 1:

$$F = F_1(\mathbf{q}, \mathbf{Q}, t) \qquad p_i = +\frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$$

Case 2:

$$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q}\cdot\mathbf{P} \qquad p_i = +\frac{\partial F_2}{\partial q_i}, \quad Q_i = +\frac{\partial F_2}{\partial P_i}$$

Case 3:

$$F = F_3(\mathbf{Q}, \mathbf{p}, t) + \mathbf{q}\cdot\mathbf{p} \qquad q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

Case 4:

$$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q}\cdot\mathbf{p} - \mathbf{Q}\cdot\mathbf{P} \qquad q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = +\frac{\partial F_4}{\partial P_i}$$

- In all cases, new Hamiltonian and equations of motion become:

$$K = H + \frac{\partial F_i}{\partial t}, \qquad \frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i}, \qquad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i}$$

[Review] An Example of the Generation Function

- For F_3 we will show

$$\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t) = \mathbf{P}\cdot\dot{\mathbf{Q}} - K(\mathbf{Q},\mathbf{P},t) + \frac{dF}{dt}$$

- Proof:

$$F = F_3(\mathbf{Q},\mathbf{p},t) + \mathbf{q}\cdot\mathbf{p}$$

$$\begin{aligned}
\frac{dF}{dt} &= \frac{\partial F_3}{\partial t} + \sum_i \left(\underbrace{\frac{\partial F_3}{\partial Q_i}}_{=-P_i}\dot{Q}_i + \underbrace{\frac{\partial F_3}{\partial p_i}}_{=-q_i}\dot{p}_i + +q_i\dot{p}_i + p_i\dot{q}_i \right) \\
&= \frac{\partial F_3}{\partial t} + \sum_i \left(-P_i\dot{Q}_i - \cancel{q_i\dot{p}_i} + \cancel{q_i\dot{p}_i} + p_i\dot{q}_i \right) \\
&= \frac{\partial F_3}{\partial t} - \mathbf{P}\cdot\dot{\mathbf{Q}} + \mathbf{p}\cdot\dot{\mathbf{q}}
\end{aligned}$$

Therefore

$$K = H + \frac{\partial F_3}{\partial t} \longrightarrow \mathbf{p}\cdot\dot{\mathbf{q}} - H = \mathbf{P}\cdot\dot{\mathbf{Q}} - K + \frac{dF}{dt}$$

Generation Function of Type 2

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- Old variables are $(z, p_{z,c})$ and the new variables are $(\zeta, p_{\zeta,c})$
- If we introduce a type 2 generating function as $F_2(z, p_{\zeta,c}, t) = (z - v_\phi t) p_{\zeta,c}$, from the canonical transformation,

$$p_i = +\frac{\partial F_2}{\partial q_i} \longrightarrow \quad p_{z,c} = \frac{\partial F_2}{\partial z} = p_{\zeta,c}$$

$$Q_i = +\frac{\partial F_2}{\partial P_i} \longrightarrow \quad \zeta = \frac{\partial F_2}{\partial p_{\zeta,c}} = z - v_\phi t$$

- Then the new Hamiltonian becomes

$$K(\text{or } \tilde{H}) = H + \frac{\partial F_2}{\partial t} = H - v_\phi p_{\zeta,c}$$

$$\begin{aligned} \tilde{H}(\zeta, p_{\zeta,c}) &= H(\zeta, p_{\zeta,c}) - v_\phi p_{\zeta,c} \\ &= \sqrt{\left(p_{\zeta,c} + \frac{qE_0}{k_z v_\phi} \cos[k_z \zeta]\right)^2 c^2 + (m_0 c^2)^2} - v_\phi p_{\zeta,c} \end{aligned}$$ Eq. (4.13)

→ It is clear that the new Hamiltonian is in fact a constant of the motion

New Equations of Motion

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- The equations of motion derived from the new Hamiltonian are thus

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \frac{\partial \tilde{H}}{\partial p_{\zeta,c}} = \frac{p_{\zeta,c} + (qE_0/k_z v_\varphi)cos[k_z\zeta]}{\gamma m_0} - v_\varphi = \frac{p_\zeta}{\gamma m_0} - v_\varphi = v_z - v_\varphi, \quad (4.14)$$

$$\frac{\mathrm{d}p_{\zeta,c}}{\mathrm{d}t} = -\frac{\partial \tilde{H}}{\partial \zeta} = \frac{qE_0 v_z}{v_\varphi}\sin[k_z(\zeta)], \quad (4.15)$$

- More convenient form: Once we find the correct Hamiltonian and corresponding equations of motions, we can revert Eq. (4.13) to the mechanical description. It is more convenient to visualize the motion of the charged particle in the longitudinal phase space $(\zeta, p_\zeta = \gamma m_0 v_z)$ $= p_z$ $\neq \gamma m_0 d\zeta/dt$

$$\tilde{H}(\zeta, p_\zeta) = \sqrt{p_\zeta^2 c^2 + (m_0 c^2)^2} - v_\varphi p_\zeta + \frac{qE_0}{k_z}\cos[k_z\zeta]. \quad (4.17)$$

- In the normalized form:

$$\frac{\tilde{H}}{m_0 c^2} = \sqrt{(\beta_z\gamma)^2 + 1} - \beta_\phi \beta_z \gamma + \alpha_{\mathrm{rf}}\cos[k_z\zeta].$$

$$\alpha_{\mathrm{rf}} \equiv \frac{qE_0}{k_z m_0 c^2} = \frac{\gamma'_{\max}}{k_z}$$

See Eq. (2.29): the ratio of the maximum spatial rate of change of the normalized particle energy to the maximum spatial rate of change of the particle's phase in the wave

Violent Accelerating System ($\alpha_{rf} \geq 1$)

- Violent acceleration: Particle can gain more than one unit of rest energy by remaining in synchronism with the wave for a radian or less of spatial propagation.

$$\alpha_{rf} = \frac{\frac{qE_0}{m_0c^2}[1/\mathrm{m}]}{k_z[1/\mathrm{m}]} = \frac{\frac{\Delta(qV)}{m_0c^2}/\Delta z}{2\pi/\lambda} = \frac{\frac{\Delta(qV)}{m_0c^2}}{(2\pi/\lambda)\Delta z} = \frac{\text{Energy gain in the unit of rest energy}}{\text{Spatial propagation in the unit of radian}}$$

$$\alpha_{rf} \geq 1$$

- Also we assume that the phase velocity of the wave reaches its ultra-relativistic limit. If the v_ϕ is chosen to be noticeably less than c, the particles can accelerate past this phase velocity, and eventually outrun the wave to the point where they may enter a decelerating phase.

$$v_\phi \approx c$$

- With these approximations,

$$\frac{\tilde{H}}{m_0c^2} = \sqrt{(\beta_z\gamma)^2+1} - \beta_\phi\beta_z\gamma + \alpha_{\mathrm{rf}}\cos[k_z\zeta]. \qquad (4.18)$$

$$\rightarrow \quad \sqrt{(\beta_z\gamma)^2+1} = \sqrt{\frac{\beta_z^2}{1-\beta_z^2}+1} = \gamma, \gamma - \beta_z\gamma = \sqrt{\frac{1-\beta_z}{1+\beta_z}}$$

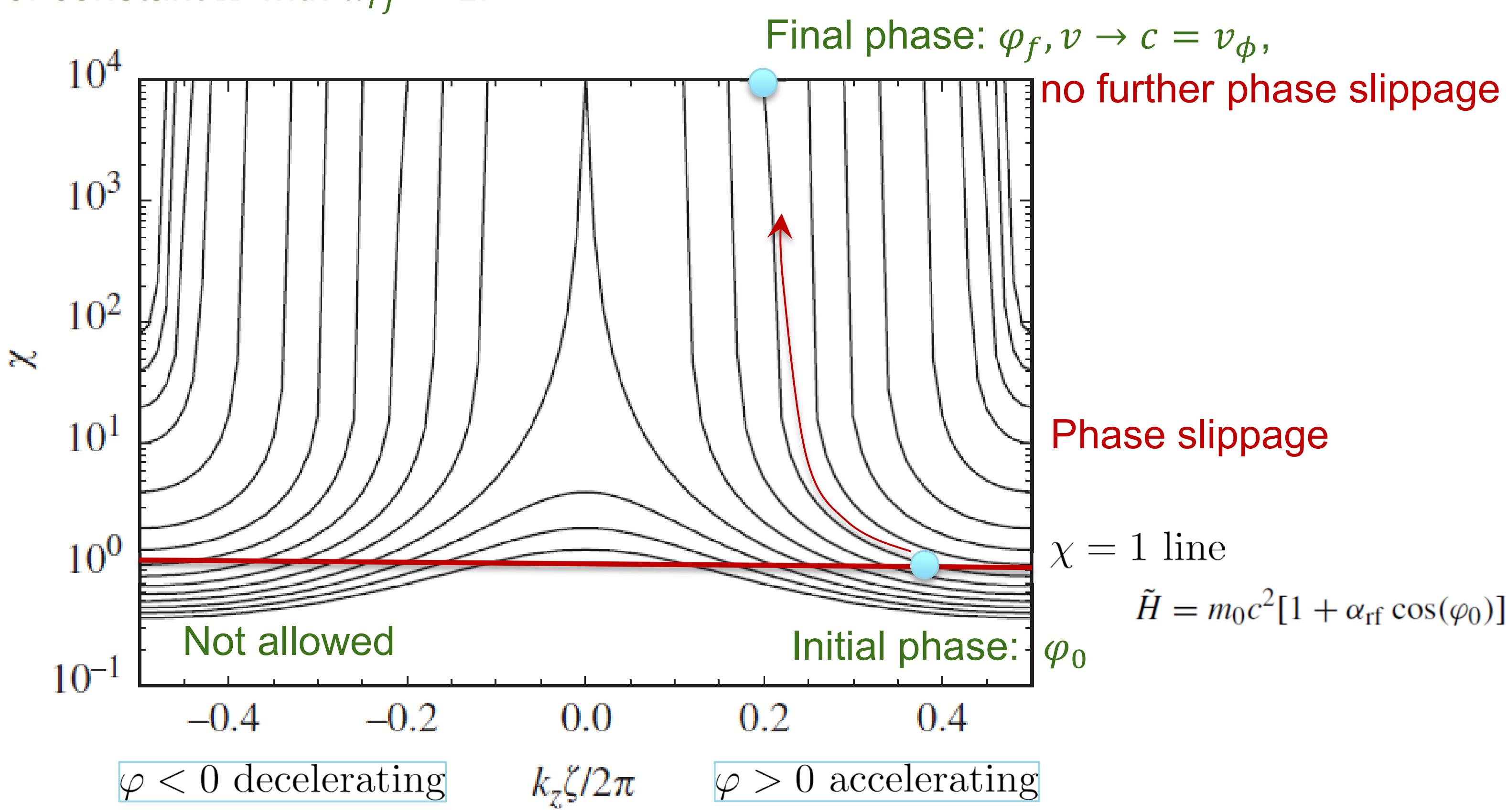
$$\tilde{H}(\zeta, p_\zeta) = m_0c^2[\gamma - \beta_z\gamma + \alpha_{\mathrm{rf}}\cos[k_z\zeta]] = m_0c^2\left[\sqrt{\frac{1-\beta_z}{1+\beta_z}} + \alpha_{\mathrm{rf}}\cos[k_z\zeta]\right]. \qquad (4.20)$$

Violent Accelerating System ($\alpha_{rf} \geq 1$)

- To visualize the dynamics of the accelerating process in the phase plane, we define a new parameter

$$\chi = \sqrt{\frac{1+\beta_z}{1-\beta_z}} \quad \begin{cases} \beta_z = 0, & \chi = 1 \\ \beta_z = 1, & \chi \to \infty, \end{cases}$$

- Contours of constant $\tilde{H}$ with $\alpha_{rf} = 1$:



Bottom left case in slide 52

Gentle Accelerating Systems ($\alpha_{rf} \ll 1$)

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- For heavy particles (proton, heavy ions), one always finds $\alpha_{rf} \ll 1$. For these gentle accelerating systems, the energy gain over a wavelength of the acceleration is much less than the rest mass.

- The motion in these systems is characterized by simple harmonic motion near the stable fixed point.

- The design (reference) momentum is given by

$$p_0 = \gamma_0 m_0 v_0 = \frac{m_0 v_0}{\sqrt{1-(v_0/c)^2}} = \frac{m_0 v_\varphi}{\sqrt{1-(v_\varphi/c)^2}} = \gamma_\varphi m_0 v_\varphi$$

  – Particle is resonant with the phase velocity of the wave

- Expanding up to second order in $\delta p = p_\zeta - p_0$

$$\begin{aligned} c\sqrt{(\delta p + p_0)^2 + (m_0^2 c^2)} &= c\sqrt{\delta p^2 + 2\delta p p_0 + p_0^2 + m_0^2 c^2} = c\sqrt{p_0^2 + m_0^2 c^2}\sqrt{1 + \frac{\delta p^2 + 2\delta p p_0}{p_0^2 + m_0^2 c^2}} \\ &\simeq \gamma_0 m_0 c^2 \left[1 + \frac{1}{2}\frac{\delta p^2 + 2\delta p p_0}{p_0^2 + m_0^2 c^2} - \frac{1}{8}\left(\frac{\delta p^2 + 2\delta p p_0}{p_0^2 + m_0^2 c^2}\right)^2\right] \\ &\simeq \gamma_0 m_0 c^2 \left[1 + \frac{1}{2}\frac{\delta p^2 + 2\delta p p_0}{\gamma_0^2 m_0^2 c^2} - \frac{1}{2}\frac{\delta p^2 p_0^2}{(\gamma_0^2 m_0^2 c^2)^2}\right] = \gamma_0 m_0 c^2 + v_0 \delta p + \frac{\delta p^2}{2\gamma_0^3 m_0} \end{aligned}$$

Gentle Accelerating Systems ($\alpha_{rf} \ll 1$)

- We have the following expression

$$\tilde{H}(\zeta, p_\zeta) = \sqrt{p_\zeta^2 c^2 + (m_0 c^2)^2} - v_\varphi p_\zeta + \frac{qE_0}{k_z}\cos[k_z \zeta]. \qquad (4.17)$$

$$\begin{aligned} \gamma_0 m_0 c^2 - v_0 p_0 &= \gamma_0 m_0 c^2 - \gamma m_0 v_0^2 \\ &= \gamma_0 m_0 c^2 \left(1 - \frac{v_0^2}{c^2}\right) \\ &= \frac{m_0 c^2}{\gamma_0} \end{aligned}$$

$$\begin{aligned} \tilde{H}(\zeta, \delta p) &\cong \gamma_0 m_0 c^2 + v_0 \delta p + \frac{\delta p^2}{2\gamma_0^3 m_0} - v_0(p_0 + \delta p) + \frac{qE_0}{k_z}\cos(k_z \zeta) \\ &= \frac{m_0 c^2}{\gamma_0} + \frac{\delta p^2}{2\gamma_0^3 m_0} + \frac{qE_0}{k_z}\cos(k_z \zeta). \end{aligned} \qquad (4.26)$$

- The addition and subtraction of constants in the Hamiltonian have no effect on the form of the phase plane curves, or on the derived equations of motion.

$$\tilde{H}(\zeta, \delta p) = (m_0 c^2)\left[\frac{\beta_0^2}{2\gamma_0 p_0^2}(\delta p^2) + \alpha_{\mathrm{rf}}[\cos(k_z \zeta) + 1]\right]. \qquad (4.27)$$

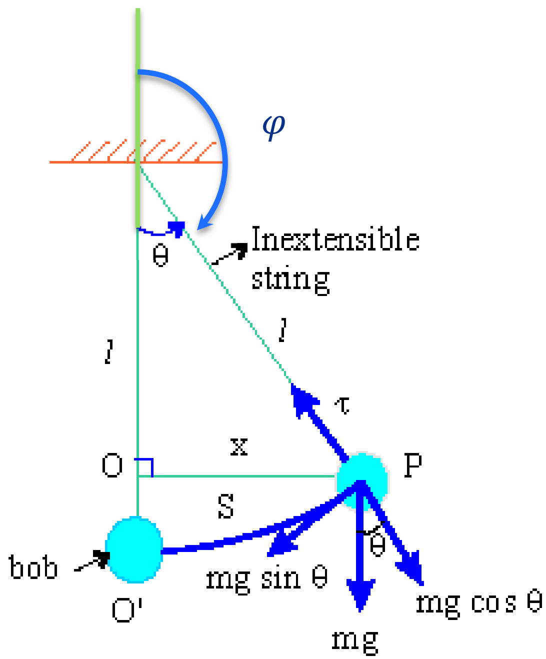
  - We can extract the equations of motion:

$$\dot{\zeta} = \frac{\partial \tilde{H}}{\partial(\delta p)} = \frac{m_0(\beta_0 c)^2}{\gamma_0 p_0^2}\delta p = \frac{\delta p}{\gamma_0^3 m_0}, \qquad \delta\dot{p} = -\frac{\partial \tilde{H}}{\partial \zeta} = \alpha_{\mathrm{rf}} k_z m_0 c^2 \sin(k_z \zeta) = qE_0 \sin(k_z \zeta). \qquad (4.29)$$

Gentle Accelerating Systems ($\alpha_{rf} \ll 1$)

$$\tilde{H}(\zeta, \delta p) = (m_0 c^2)\left[\frac{\beta_0^2}{2\gamma_0 p_0^2}(\delta p^2) + \alpha_{\text{rf}}[\cos(k_z\zeta) + 1]\right]. \qquad (4.27)$$

- It can be seen that the Hamiltonian in Eq. (4.27) is of the from corresponding to a pendulum, where the minimum potential of the pendulum is chosen as $\varphi_{min} = \pi$



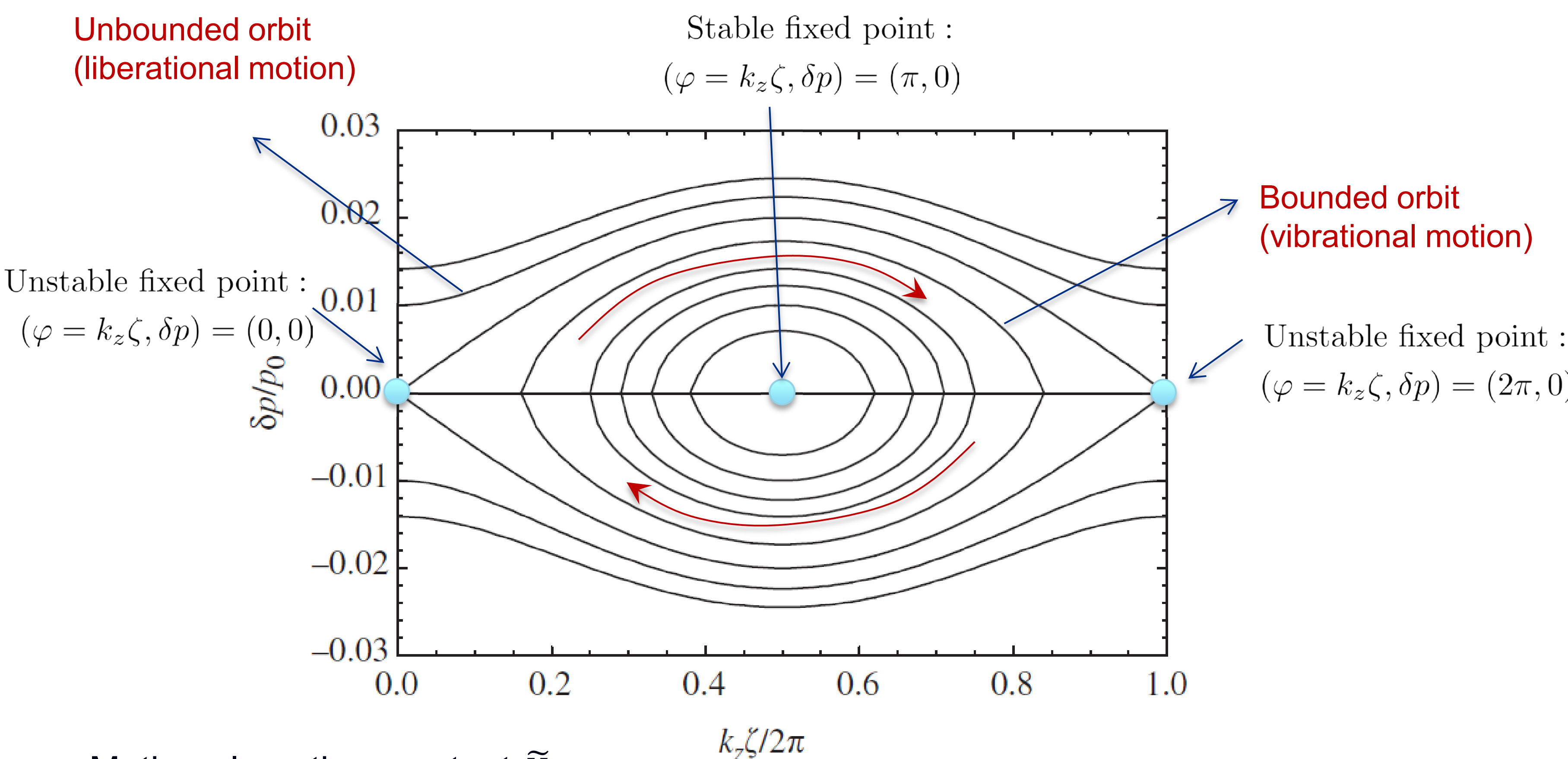
$0 < \varphi < 2\pi$

$$\text{KE} = \frac{1}{2}m(l\dot{\varphi})^2$$

$$\begin{aligned}\text{PE} &= mg\,[l - l\cos(\theta)] = mg\,[(l - l\cos(\pi - \varphi)]\\ &= mg\,[l + l\cos(\varphi)]\end{aligned}$$

Gentle Accelerating Systems ($\alpha_{rf} \ll 1$)

- Phase plane trajectories showing the stable region (bucket) of vibrational motion, bounded by a separatrix:



- Motion along the constant $\widetilde{H}$ curves:

$$\text{For } \delta p > 0 \;\rightarrow\; \text{towards positive } \zeta$$

$$\text{For } \delta p < 0 \;\rightarrow\; \text{towards negative } \zeta$$

Gentle Accelerating Systems ($\alpha_{rf} \ll 1$)

- The equation for the separatrix:

$$\tilde{H}(\zeta, \delta p) = (m_0 c^2)\left[\frac{\beta_0^2}{2\gamma_0 p_0^2}(\delta p^2) + \alpha_{rf}[\cos(k_\zeta \zeta) + 1]\right] = \tilde{H}(0,0) = 2\alpha_{rf} m_0 c^2$$

$$\frac{\delta p_{\mathrm{sep}}}{p_0} = \pm\frac{1}{\beta_0}\sqrt{2\alpha_{\mathrm{rf}}\gamma_0[1-\cos(k_z\zeta)]} = \pm\sqrt{\frac{4\alpha_{\mathrm{rf}}\gamma_0}{\beta_0^2}}\sin\left(\frac{k_z\zeta}{2}\right). \qquad (4.31)$$

- The peak momentum offset encountered in the bucket occurs at $k_z\zeta = \pi$.

$$\frac{\delta p_{\max}}{p_0} = \pm\sqrt{\frac{4\alpha_{\mathrm{rf}}\gamma_0}{\beta_0^2}}.$$

- Since the particles are moderately relativistic ($\gamma_0$is not many orders of magnitude larger than unity) and $\alpha_{rf} \ll 1$, we note that

$$\left|\frac{\delta p_{max}}{p_0}\right| = \sqrt{\frac{4\alpha_{rf}\gamma_0}{\beta_0^2}} \ll 1$$

- The area of the stable phase plane $(\zeta, \delta p)$: Bucket area

$$A_b = 2p_0\sqrt{\frac{4\alpha_{rf}\gamma_0}{\beta_0^2}}\int_0^{2\pi/k_z}\sin(k_z\zeta/2)d\zeta = \frac{16p_0}{k_z}\sqrt{\frac{\alpha_{rf}\gamma_0}{\beta_0^2}}$$

Gentle Accelerating Systems ($\alpha_{rf} \ll 1$)

- Even though this large amplitude motion (with its nonlinear characteristics) is unfamiliar, the small amplitude motion about the stable fixed point is quite familiar. If we expand the Hamiltonian near this point, we have

$$\delta\zeta = \zeta - \pi/k_z$$

$$\cos(k_z\zeta) = \cos[k_z(\delta\zeta + \pi/k_z)] = -\cos[k_z\delta\zeta] \simeq -1 + \frac{1}{2}(k_z\delta\zeta)^2$$

$$\tilde{H}(\zeta, \delta p) \cong (m_0c^2)\left[\frac{\beta_0^2}{2\gamma_0 p_0^2}(\delta p^2) + \frac{\alpha_{\rm rf}(k_z\delta\zeta)^2}{2}\right], \qquad (4.34)$$

- This small amplitude Hamiltonian can be used to obtain the simple harmonic oscillator equation:

$$\dot{\delta\zeta} = \frac{\partial\tilde{H}}{\partial(\delta p)} = \frac{m_0c^2\beta_0^2}{\gamma_0 p_0^2}\delta p = \frac{\delta p}{\gamma_0^3 m_0}, \quad \dot{\delta p} = -\frac{\partial\tilde{H}}{\partial(\delta\zeta)} = -m_0c^2\alpha_{rf}k_z^2\delta\zeta$$

$$\delta\ddot{\zeta} + \frac{\alpha_{\rm rf}(k_z c)^2}{\gamma_0^3}\delta\zeta = 0. \qquad (4.35)$$

- Equation (4.35) gives solution termed synchrotron oscillations, that are harmonic with the synchrotron frequency:

$$\omega_s = k_z c\sqrt{\frac{\alpha_{\rm rf}}{\gamma_0^3}} = \sqrt{\underbrace{\frac{\alpha_{\rm rf}}{\gamma_0^3}}_{\ll 1}}\,\frac{\omega}{\beta_0}.$$

The synchrotron frequency is much smaller than the frequency of the wave

The Moving Bucket

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- For the gentle accelerating system, stable buckets do not allow significant acceleration.

- By slowly increasing the design velocity (or synchronous velocity) v_0, we can make on average all of the particles in the moving bucket gain energy.

$$\frac{dv_0}{dz} = \frac{1}{v_0}\frac{dv_0}{dt} = \frac{F_z(z)}{\gamma_0^3 m_0 v_0} = \frac{qE_0(z)\sin(\varphi_0(z))}{\gamma_0^3 m_0 v_0}$$

Changing phase different from π

  - Here, v_0 is no longer constant in space.

- To satisfy the synchronous condition, $v_0(z) = v_\phi = \dfrac{\omega}{k_z(z)}$

  - The accelerating frequency is held constant, but the spatial periodicity is changing.
  - So $k_z(z)$ is a decreasing function of distance.

- [Note] Currently, we're discussing the case in the linac. In circular accelerators, where the periodicity in space (set by the circumference of the accelerator) is constant, the synchronous velocity, $v_0(t) = \omega(t)/k_z$ is raised by increasing the frequency of the applied accelerating field in a localized accelerating structure

The Moving Bucket

- The acceleration of the synchronous particle means that the entire reference frame is accelerated in the forward direction. This yields an effective uniform force in the reverse direction (like the force one feels in an accelerating vehicle; inertial force),

$$F_{z,eff} = -qE_0 \sin(\varphi_0)$$

- This force can be included in a new Hamiltonian as an accelerating potential:

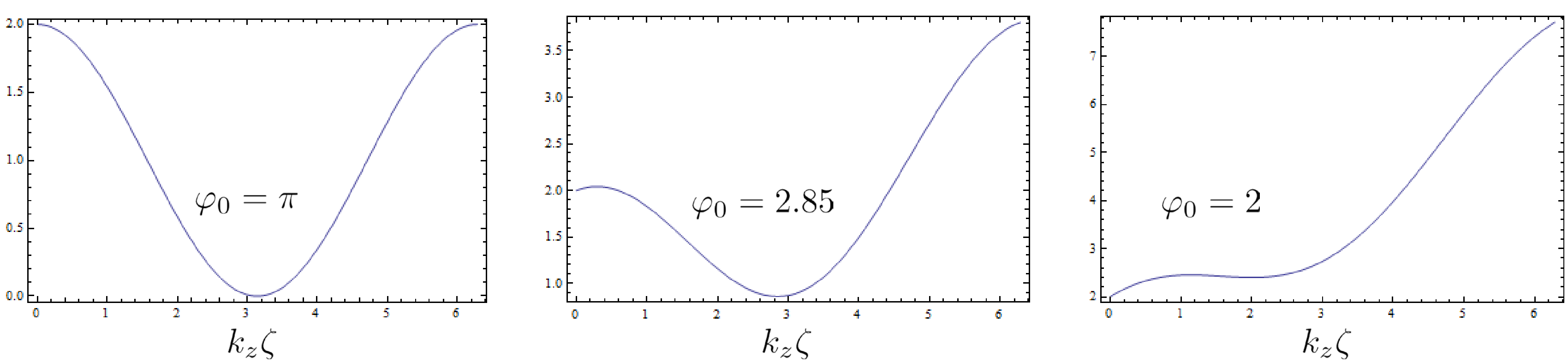
$$\hat{H}(\zeta, \delta p) = (m_0 c^2)\left[\frac{\beta_0^2}{2\gamma_0 p_0^2}(\delta p)^2 + \underbrace{\alpha_{rf}[\cos(k_z\zeta) + k_z\zeta\sin(\varphi_0) + 1]}_{\text{Effective potential}}\right]$$

Effective potential

$$\dot{\delta p} = -\frac{\partial \hat{H}}{\partial \zeta} = m_0 c^2 \alpha_{rf} k_z \sin(k_z\zeta) - m_0 c^2 \alpha_{rf} k_z \sin(\varphi_0) = qE_0 \sin(k_z\zeta) \underbrace{- qE_0 \sin(\varphi_0)}_{\text{Effective force}}$$

Effective force

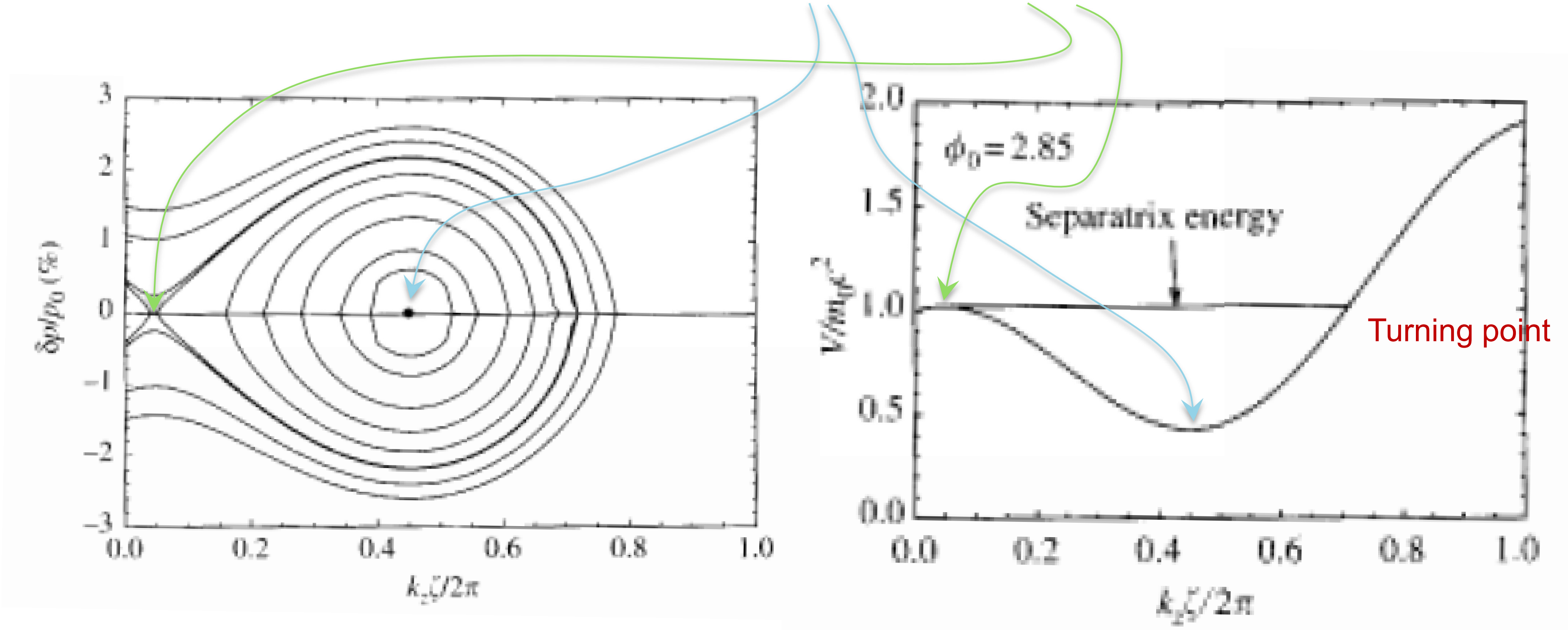
- Plots of the effective potential:



The Moving Bucket

- The derivative of the effective potential with respect to $k_z\zeta$ gives two fixed points.

$$-\sin(k_z\zeta)+\sin(\varphi_0)=0 \longrightarrow k_z\zeta=\varphi_0 \quad \text{or} \quad k_z\zeta=\pi-\varphi_0$$



- Equation for the separatrix: $\hat{H}(k_z\zeta=\pi-\varphi_0,\delta p=0)=\alpha_{rf}m_0c^2[1-\cos(\varphi_0)+(\pi-\varphi_0)\sin(\varphi_0)]$

- Phase at the turning point: $\cos(\varphi_{turn})+\varphi_{turn}\sin(\varphi_0)=1-\cos(\varphi_0)+(\pi-\varphi_0)\sin(\varphi_0)$

The Moving Bucket

- The fixed point at φ_0 has stable synchrotron oscillations about it. Using the following expansion in the Hamiltonian,

$$\delta\zeta = \zeta - \varphi_0/k_z$$

$$\begin{aligned} \cos(k_z\delta\zeta + \varphi_0) + (k_z\delta\zeta + \varphi_0)\sin(\varphi_0) &= \cos(k_z\delta\zeta)\cos(\varphi_0) - \sin(k_z\delta\zeta)\sin(\varphi_0) \\ &+ \sin(k_z\delta\zeta)\sin(\varphi_0) + \varphi_0\sin(\varphi_0) \end{aligned}$$

Constant term

- A new simple harmonic oscillator

$$\ddot{\delta\zeta} - \frac{\alpha_{rf}(k_z c)^2}{\gamma_0^3}\cos(\varphi_0)\delta\zeta = 0$$

with smaller value of the synchronous frequency for $\pi/2 < \varphi_0 < \pi$

$$\omega_s = \sqrt{\frac{\alpha_{rf}}{\gamma_0^3}|\cos(\varphi_0)|} \times \frac{\omega}{\beta_0}$$

Acceleration in Circular Machines

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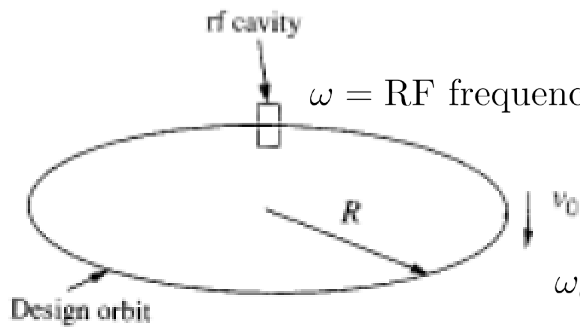
- To reuse the RF cavity in the accelerating process many times:

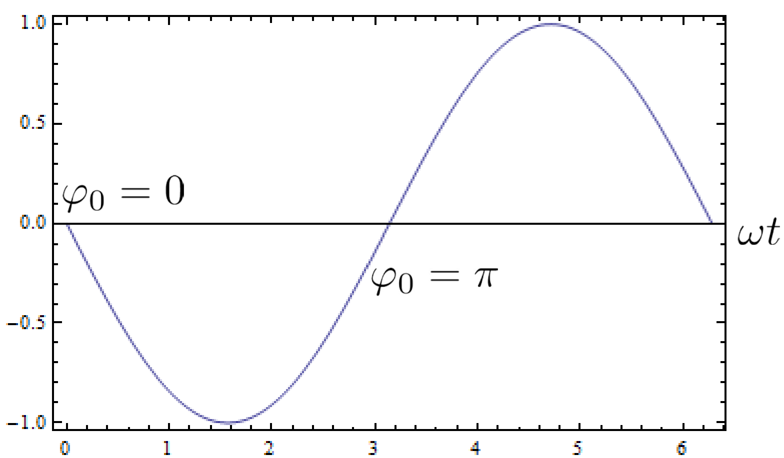


Harmonic number

$\omega = \text{RF frequency} = h\omega_c$

$\omega_c = \text{Circulation or revolution frequency}$

- Longitudinal stability or phase-focusing is provided by use of the time varying fields inside the cavity.



- Looking at the motion at only on position in the ring (i.e., RF cavity with negligible length):

$$V_0 \sin(k_z z - \omega t) \longrightarrow -V_0 \sin(\omega t)$$

Travelling wave → Standing wave

- We postulate the existence of a particle on the design orbit, with constant design velocity $v_0 = R\omega_c$, and with constant phase $\varphi_0 = \pi$ as in the case of stationary bucket in the linac (assuming no net energy gain/loss).

Acceleration in Circular Machines

- We construct the dynamics “turn by turn”. The energy of a particle after its (n+1) traversal of the RF cavity is related to its energy on the previous turn by

$$\delta U_{n+1} = \delta U_n - qV_0 \sin(\omega\tau_n + \pi) = \delta U_n + qV_0 \sin(\omega\tau_n)$$

  - where τ is the time of arrival of the particle at the RF cavity with respect to the arrival of the design particle,
  - and, $\delta U = U - U_0$ is the difference in particle energy from the design value.

- From the definition of the phase slip factor (slide 10 of Lecture 4)

$$\frac{\delta\tau}{t_0} \simeq \left[\alpha_c - \frac{1}{\gamma_0^2}\right]\frac{\delta p}{p_0} \equiv \eta_\tau \frac{\delta p}{p_0}$$

$$t_0 = \frac{2\pi}{\omega_c} = \text{Turn-by-Turn revolution time}, \quad \frac{\delta p}{p_0} = \frac{1}{\beta_0^2}\frac{\delta U}{U_0}$$

- We examine the change in time of arrival “turn by turn”.

$$\tau_{n+1} = \tau_n + \delta\tau_n = \tau_n + t_0\eta_\tau \frac{\delta p_n}{p_0} = \frac{2\pi\eta_\tau}{\omega_c}\frac{\delta p_n}{p_0} = \frac{2\pi\eta_\tau}{\omega_c\beta_0^2}\frac{\delta U_n}{U_0}$$

Set of difference equations

Acceleration in Circular Machines

- The difference equations can be understood by viewing them as numerically equivalent to the differential equations.

$$\frac{d(\delta U)}{dt} = \lim_{t_0 \to 0} \frac{U_{n+1} - U_n}{t_0} = \lim_{t_0 \to 0} \frac{\Delta U}{t_0} = \frac{q\omega_c V_0}{2\pi} \sin(\omega\tau)$$

$$\frac{d\tau}{dt} = \lim_{t_0 \to 0} \frac{\tau_{n+1} - \tau_n}{t_0} = \lim_{t_0 \to 0} \frac{\Delta \tau}{t_0} = \frac{\eta_\tau}{\beta_0^2 U_0} \delta U$$

  – We assume that the changes in the variables are not too significant in one turn.

- The second order differential equation derived from the above expressions is

$$\frac{d^2\tau}{dt^2} - \frac{\eta_\tau \omega_c q V_0}{2\pi \beta_0^2 U_0} \sin(\omega\tau) = 0$$

- Assuming an energy below transition, and expanding near $\tau = 0$,

$$\frac{d^2\tau}{dt^2} + \frac{|\eta_\tau| h \omega_c^2 q V_0}{2\pi \beta_0^2 U_0} \tau = 0$$

- For the small amplitude oscillation, the synchrotron frequency and tune are

$$\omega_s = \omega_c \sqrt{\frac{|\eta_\tau| h q V_0}{2\pi \beta_0^2 U_0}}, \quad \nu_s = \frac{\omega_s}{\omega_c} = \sqrt{\frac{|\eta_\tau| h q V_0}{2\pi \beta_0^2 U_0}} \ll 1$$

Validity for approximating the difference equations to differential equations