Lecture 3

'Linear' Transverse Dynamics

(Ch. 3 of FOBP, Ch. 2 of UP-ALP)

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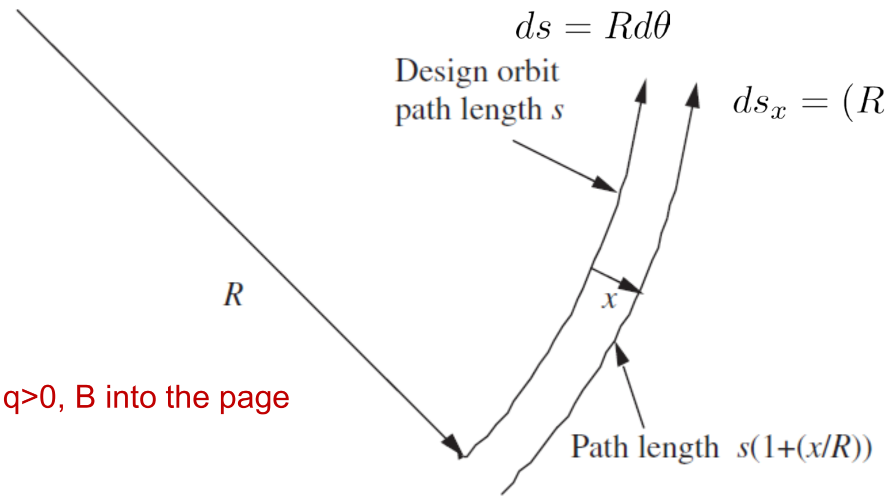
Sec. 3.1

Weak Focusing in Circular Accelerators

[Review] Path length focusing

- In Chapter 2, we learned that path length focusing is effective in stabilizing the horizontal motion (x), but not in the vertical motion (y).



$$ds_x = (R+x)d\theta = R(1+x/R)d\theta = (1+x/R)ds$$

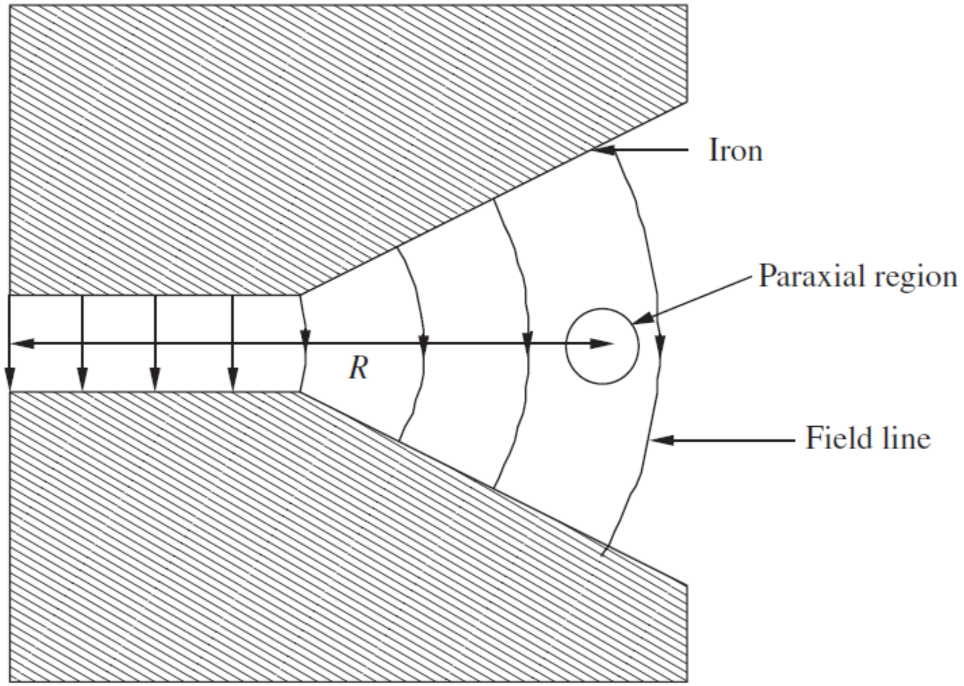
For q>0, B into the page

$$\Delta p_x = -q\int_{t_1}^{t_2} v_0 B_0 dt = -q\int_{t_1}^{t_2} B_0 ds_x = -q\int_{s_1}^{s_2} B_0\left(1+\frac{x}{R}\right)ds = -q\int_{s_1}^{s_2} B_0 ds - q\int_{s_1}^{s_2} B_0\frac{x}{R}ds$$

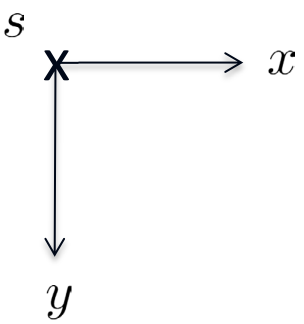
$$x'' + \left(\frac{1}{R}\right)^2 x = 0$$

Magnetic fields in betatron (β particle = fast e⁻)

- Near the design orbit:



$$B_0 R = \frac{p_0}{|q|}$$



$$B' < 0$$

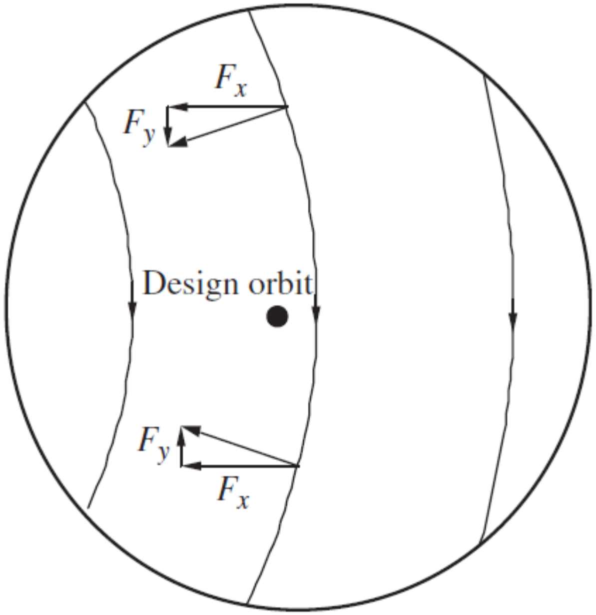
$$B_y(x) = B_0 + \frac{\partial B_y}{\partial x} x + \cdots = B_0 + B' x + \cdots$$

$$B_x(y) = \int_0^y \frac{\partial B_y}{\partial x} d\tilde{y} \simeq B' y$$

$$\nabla \times \mathbf{B} = 0$$

Equation of motion in betatron

- The magnetic field appears as a superposition of vertically oriented dipole and vertically focusing (horizontally defocusing) quadrupole fields.



Electron is coming out of the paper

From dipole components

$$B_0 R = \frac{p_0}{|q|}$$

$$x'' + \left[\left(\frac{1}{R}\right)^2 + \frac{B'}{B_0 R}\right] x = 0$$

$$y'' - \frac{B'}{B_0 R} y = 0$$

From quadrupole components

- In terms of field index: $n \equiv -\frac{B' R}{B_0}$

$$x'' + \left(\frac{1}{R}\right)^2 [1 - n] x = 0 \qquad y'' + \frac{n}{R^2} y = 0 \qquad \longrightarrow \qquad 0 < n < 1$$

For simultaneous stability

Tunes (denoted by either ν or Q)

- If we write the equations of motion in terms of azimuthal angle $\theta = s/R$:

$$x'' + \left(\frac{1}{R}\right)^2 [1-n]\, x = 0 \longrightarrow \frac{d^2x}{d\theta^2} + [1-n]x = 0 \longrightarrow \frac{d^2x}{d\theta^2} + \nu_x^2 x = 0$$

$$y'' + \left(\frac{n}{R}\right)^2 y = 0 \longrightarrow \frac{d^2y}{d\theta^2} + ny = 0 \longrightarrow \frac{d^2y}{d\theta^2} + \nu_y^2 y = 0$$

- The phase changes (or phase advances) per one period (for circular machine considered here, one revolution, 2π) are

$$\Delta\phi_x = 2\pi\nu_x, \quad \Delta\phi_y = 2\pi\nu_y$$

- The number of oscillations in the horizontal (x) and vertical (y) dimensions per one period (for circular machine considered here, one revolution, 2π) are called tunes:

$$\nu_x = \sqrt{1-n}, \quad \nu_y = \sqrt{n}$$

- Restriction on tunes for betatron (weak focusing): $\nu_x, \nu_y < 1$
- Scaling of the maximum offset→ size of the beam scales with the radius of curvature

$$x \sim x_m \sin(\nu_x/Rz + \phi_0) \quad \longrightarrow x' \sim x_m \nu_x/R \quad \longrightarrow x_m \sim Rx'/\nu_x$$

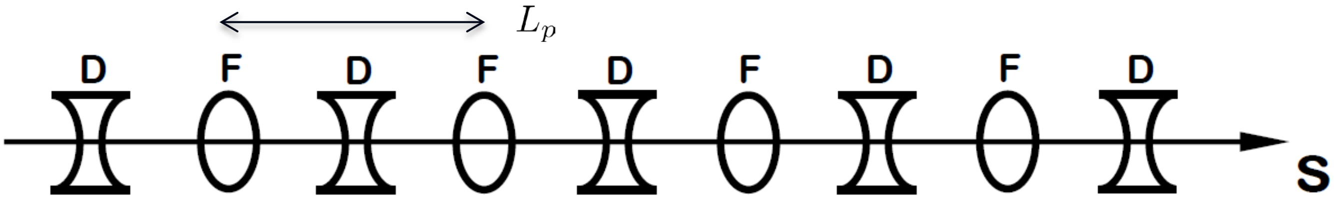
We need to make tune very large:
Strong focusing

Sec. 3.2

Matrix Analysis of Periodic Focusing System

Periodic focusing

- Most large accelerators are made up of several (or many) identical modules and, therefore, have periodicity of L_p:
  - Circular machine: $L_p = C/M_p$
  - Linear machine: array of simple quadrupole magnets with differing sign field gradient



- Hill's equation:

$$x'' + \kappa_x^2(z)x = 0, \qquad \kappa_x^2(z + L_p) = \kappa_x^2(z) \equiv K_x(z) \text{ in some other books}$$

- Two special cases which can be readily analyzed.
  - The focusing is sinusoidally varying: Mathieu equation
  - The focusing is piece-wise constant : Combination of a number of simple harmonic oscillator solutions

Matrix formalism

- Initial state vector:

$$\mathbf{x}(z_0) = \left(\begin{array}{c} x \\ x' \end{array} \right)_{z=z_0} = \left(\begin{array}{c} x_i \\ x_i' \end{array} \right) = (x \quad x_i')^T$$

- Solution of the simple harmonic oscillator for $\kappa_0^2 > 0$:

$$\begin{aligned} x(z) &= x_i \cos[\kappa_0(z-z_0)] + \frac{x_i'}{\kappa_0} \sin[\kappa_0(z-z_0)] \\ x'(z) &= -\kappa_0 x_i \sin[\kappa_0(z-z_0)] + x_i' \cos[\kappa_0(z-z_0)] \end{aligned}$$

  - If conveniently expressed by a matrix expression:

$$\mathbf{x}(z) = \mathbf{M}_F \cdot \mathbf{x}(z_0)$$

$$\mathbf{M}_F = \left[\begin{array}{cc} \cos[\kappa_0(z-z_0)] & \frac{1}{\kappa_0}\sin[\kappa_0(z-z_0)] \\ -\kappa_0 \sin[\kappa_0(z-z_0)] & \cos[\kappa_0(z-z_0)] \end{array} \right]$$

  - Though a focusing section of length l:

$$\mathbf{M}_F = \left[\begin{array}{cc} \cos[\kappa_0 l] & \frac{1}{\kappa_0}\sin[\kappa_0 l] \\ -\kappa_0 \sin[\kappa_0 l] & \cos[\kappa_0 l] \end{array} \right]$$

Matrix formalism (cont'd)

- Solution of the simple harmonic oscillator for $\kappa_0^2 = -|\kappa_0|^2 < 0$:

$$\begin{aligned} x(z) &= x_i \cosh[|\kappa_0|(z-z_0)] + \frac{x_i'}{|\kappa_0|}\sinh[|\kappa_0|(z-z_0)] \\ x'(z) &= |\kappa_0| x_i \sinh[|\kappa_0|(z-z_0)] + x_i' \cosh[|\kappa_0|(z-z_0)] \end{aligned}$$

  - If conveniently expressed by a matrix expression:

$$\mathbf{x}(z) = \mathbf{M}_D \cdot \mathbf{x}(z_0)$$

$$\mathbf{M}_D = \begin{bmatrix} \cosh[|\kappa_0|(z-z_0)] & \frac{1}{|\kappa_0|}\sinh[|\kappa_0|(z-z_0)] \\ |\kappa_0|\sinh[|\kappa_0|(z-z_0)] & \cosh[|\kappa_0|(z-z_0)] \end{bmatrix}$$

- Limiting cases:
  - Force-free drift: $\kappa_0 \to 0$

$$\mathbf{M}_F = \mathbf{M}_D = \mathbf{M}_O = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & L_d \\ 0 & 1 \end{bmatrix}$$

Length of drift space

The position x changes while the angle x' does not

  - Thin-lens limit: $l \to 0 \;\; \text{while } \kappa_0^2 l \text{ is kept finite}$

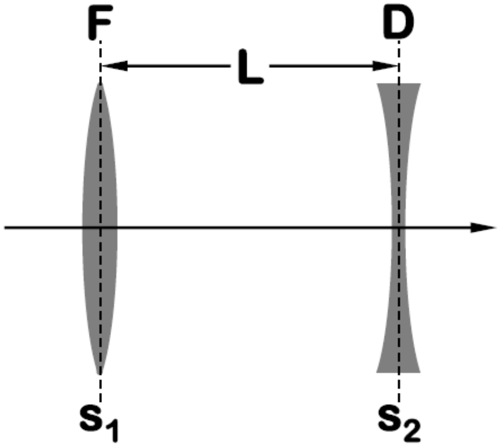
$$\mathbf{M}_{F(D)} = \begin{bmatrix} 1 & 0 \\ \mp\kappa_0^2 l & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{bmatrix}$$

The change in position x is negligible and only the angle x' is transformed

Focal length

[Example 1] Doublet

- Step-by-step matrix multiplication of all individual elements:



$$\mathrm{M}_x^{1\to 2} = \begin{bmatrix} 1 & 0 \\ \frac{1}{f_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f^*} & 1+\frac{L}{f_2} \end{bmatrix}$$

$$\frac{1}{f^*} = \frac{1}{f_1} - \frac{1}{f_2} + \frac{L}{f_1 f_2}$$

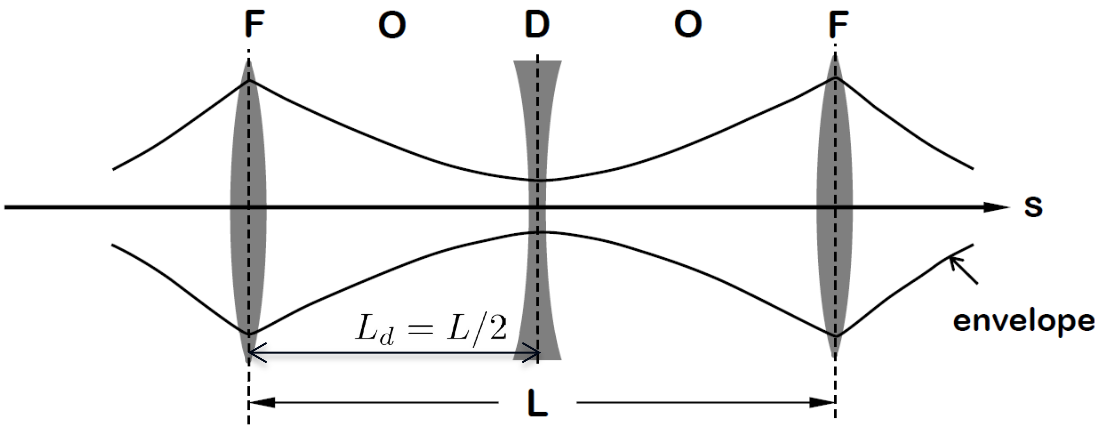
Effective focal length of the system

- For vertical direction: reversing sign of f_1 and f_2

- There is a region of parameters where the sign of f^* is the same and positive for both horizontal and vertical planes (for example, when $f_1 = f_2$), which corresponds to the focusing in both planes.

[Example 2] FODO lattice

- Focus(F)-Drift(O)-Defocus(D)-Drift(O) lattice:



$$\mathbf{x}(z) = \mathbf{x}(L + z_0) = \mathbf{x}(2L_d + 2l + z_0) = \mathbf{M}_O \cdot \mathbf{M}_D \cdot \mathbf{M}_O \cdot \mathbf{M}_F \cdot \mathbf{x}(z_0) = \mathbf{M}_T \cdot \mathbf{x}(z_0)$$

$$\mathbf{M}_T = \begin{bmatrix} 1 - \frac{L_d}{f} - \left(\frac{L_d}{f}\right)^2 & 2L_d + \frac{L_d^2}{f} \\ -\frac{L_d}{f^2} & \frac{L_d}{f} + 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial x_i} & \frac{\partial x}{\partial x'_i} \\ \frac{\partial x'}{\partial x_i} & \frac{\partial x'}{\partial x'_i} \end{bmatrix}$$

What about y direction ?

- Note that the matrix product given in Eq. (3.20) is written in reverse order from that in which the component matrices are physically encountered in the beam line. Confusion on the ordering of matrices is the most common mistake made in the matrix analysis of beam dynamics!

[Note] General properties of linear transformation

- All of the transformation matrices (the focusing, defocusing, drift, and thin lens matrices) have determinant equal to 1.

- The total transformation matrix, being the product of matrices of all of unit determinant, also has the property:

$$\det(\mathbf{M}_T) = 1$$

- The partial derivative form of the transformation matrix shows explicitly that it can be interpreted as a generalized linear transformation of coordinates in trace space.

$$\mathbf{M}_T = \begin{bmatrix} \frac{\partial x}{\partial x_i} & \frac{\partial x}{\partial x'_i} \\ \frac{\partial x'}{\partial x_i} & \frac{\partial x'}{\partial x'_i} \end{bmatrix}$$

- The determinant of this matrix is known as the Jacobian of the transformation.

- The fact that the Jacobian is unity indicates that the transformations are area preserving, as anticipated by application of Liouville's theorem.

Stability analysis

- Linear stability: Assurance of the stability of particle motion under forces that are linear in displacement from the design orbit is a necessary, but not sufficient, condition for absolutely stable motion (→ Nonlinear forces may also cause unstable orbits).

- We consider the transformation corresponding to n repeated passes through the system:

$$\mathbf{x}(nL_p + z_0) = \mathbf{M}_T^n \cdot \mathbf{x}(z_0)$$

- Eigenvector analysis:

$$\mathbf{M}_T \cdot \mathbf{d}_j = \lambda_j \mathbf{d}_j \qquad \mathbf{d}_i^T \cdot \mathbf{d}_j = \delta_{ij} \qquad \mathbf{x}(z_0) = \sum_i a_i \mathbf{d}_i, \quad \text{where} \quad a_j = \mathbf{x}^T(z_0) \cdot \mathbf{d}_j$$

- The transformation can be written in terms of eigenvectors:

$$\mathbf{x}(L_p + z_0) = \mathbf{M}_T \cdot \mathbf{x}(z_0) = a_1\lambda_1\mathbf{d}_1 + a_2\lambda_2\mathbf{d}_2$$

$$\mathbf{x}(nL_p + z_0) = \mathbf{M}_T^n \cdot \mathbf{x}(z_0) = a_1\lambda_1^n\mathbf{d}_1 + a_2\lambda_2^n\mathbf{d}_2$$

In this case,
eigenvectors are complex as well

- The eigenvalues of the transformation must be complex numbers of unit magnitude, otherwise the motion will be exponential, meaning either unbounded or decaying.

$$|\lambda_j| = 1$$

Eigenvalue problem

- Requiring the determinant of the matrix operating on the eigenvector vanish:

$$(\mathbf{M}_T - \lambda_j \mathbf{I}) \cdot \mathbf{d}_j = 0 \longrightarrow |\mathbf{M}_T - \lambda_j \mathbf{I}| = 0$$

$$\lambda_j^2 - \underbrace{(M_{T11} + M_{T22})}_{=\mathrm{Tr}(\mathbf{M}_T)} \lambda_j + \underbrace{(M_{T11}M_{T22} - M_{T12}M_{T21})}_{=\det(\mathbf{M}_T)=1} = 0$$

- For the stable motion, the eigenvalue is of unit magnitude. Hence, we choose to write the eigenvalue as (with μ being real)

$$\lambda_j = \exp(\pm i\mu)$$

- Then the solution becomes

$$\lambda_j = \exp(\pm i\mu) = \cos(\mu) \pm i \sin(\mu) = \frac{\mathrm{Tr}(\mathbf{M}_T)}{2} \pm i\sqrt{1 - \left(\frac{\mathrm{Tr}(\mathbf{M}_T)}{2}\right)^2}$$

$$2\cos(\mu) = \mathrm{Tr}(\mathbf{M}_T)$$

Stability condition

- If the term inside the square root is non-negative, the motion will be stable.

$$|\mathrm{Tr}(\mathbf{M}_T)| = |M_{T11} + M_{T22}| = |\lambda_1 + \lambda_2| \leq 2$$

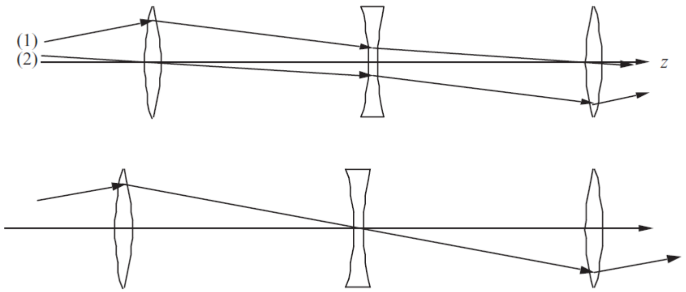
[Example] For FODO lattice

$$|\mathrm{Tr}(\mathbf{M}_T)| = |M_{T11} + M_{T22}| = \left|2 - \left(\frac{L_d}{f}\right)^2\right| \leq 2 \longrightarrow \frac{L_d}{f} = L_d(\kappa_0^2 l) \leq 2$$

$$\cos(\mu) = \frac{\mathrm{Tr}(\mathbf{M}_T)}{2} = 1 - \frac{1}{2}\left(\frac{L_d}{f}\right)^2$$

- Note:
  - We remark that since the eigenvalues of stable motion are complex, the eigenvectors are generally complex.
  - However, the transformation matrix itself is real.

- Physical meaning of μ: Phase advance per one period.



$$\mu = \frac{\pi}{2}$$

$$\mu = \pi$$

$\mu > \pi \rightarrow$ physically meaningless

Parametrization of the transformation matrix

- For stable motion,

$$\mathbf{M}_T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$a + b = \mathrm{Tr}(\mathbf{M}_T) = 2\cos(\mu), \quad \det(\mathbf{M}_T) = ad - bc = 1$$

→ From total 5 variables (a, b, c, d, μ), only 3 variables are independent

- Therefore, we may set for some real k

$$a = \cos(\mu) + k, \quad d = \cos(\mu) - k, \quad ad = \cos^2(\mu) - k^2, \quad bc = -k^2 - \sin^2(\mu)$$

- For $\sin(\mu) \neq 0$, k may be replaced by $k = \alpha \sin(\mu)$ for some real α:

$$bc = -k^2 - \sin^2(\mu) = -(1+\alpha^2)\sin^2(\mu) \longrightarrow b = \beta \sin(\mu), \quad c = -\gamma \sin(\mu)$$

- The relation between α, β, γ:

$$\beta\gamma = 1 + \alpha^2, \quad \gamma = \frac{1+\alpha^2}{\beta}$$

*Don’t be confused with relativistic factors

Parametrization of the transformation matrix

- Thus the transformation matrix (or transfer matrix) can be written as

$$
\begin{aligned}
\mathbf{M}_T &= \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cos\mu + \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\sin\mu \\
&\equiv \mathbf{I}\cos\mu + \mathbf{J}\sin\mu
\end{aligned}
$$

→ 3 independent variables

$$
\begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix} = \begin{bmatrix} \alpha^2 - \beta\gamma & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & -\beta\gamma + \alpha^2 \end{bmatrix}
$$

- Since $\mathbf{J}^2 = -\mathbf{I}$, one can apply Euler's formula for matrices:

$$\mathbf{M}_T = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu = e^{\mathbf{J}\mu}$$

- We can also obtain the De Moivere's theorem

$$\mathbf{M}_T^k = (\mathbf{I}\cos\mu + \mathbf{J}\sin\mu)^k = e^{\mathbf{J}k\mu} = \mathbf{I}\cos(k\mu) + \mathbf{J}\sin(k\mu)$$

$$\mathbf{M}_T^{-1} = (\mathbf{I}\cos\mu + \mathbf{J}\sin\mu)^{-1} = e^{-\mathbf{J}\mu} = \mathbf{I}\cos\mu - \mathbf{J}\sin\mu$$

Parametrization of the transformation matrix

- The transformation matrix can also be decomposed as

$$
\begin{aligned}
\mathbf{M}_T &= \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{bmatrix} \times \begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{bmatrix} \\
&= \mathbf{B} \begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix} \mathbf{B}^{-1}
\end{aligned}
$$

Inverse transformation into the original phase-space coordinates.

Clock-wise rotation in the normalized phase-space coordinates.

Transformation into normalized phase-space coordinates.

- Be careful! So far we only consider transfer matrix for a system with a repetitive period.

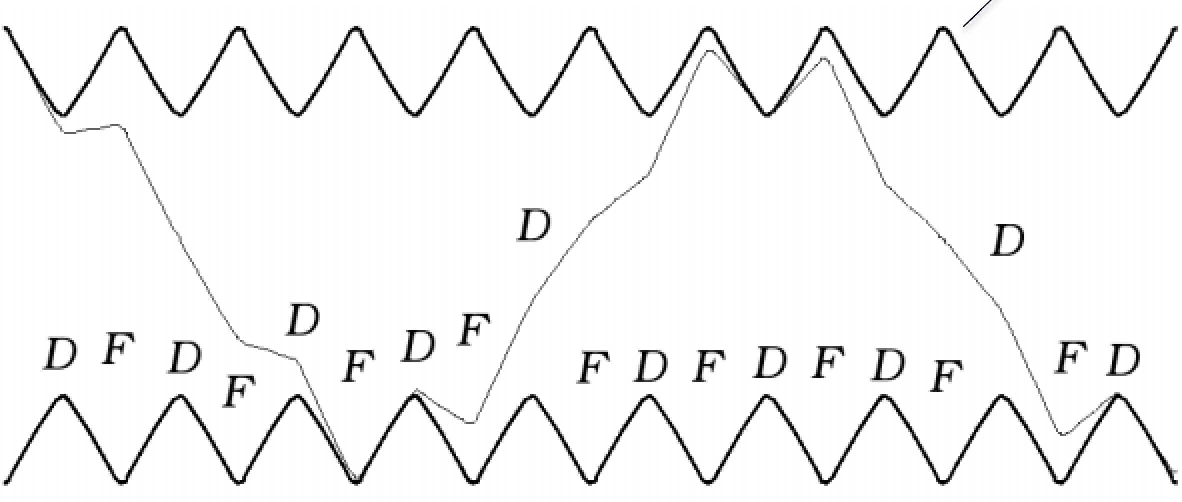
- Be careful! The $\alpha, \beta, \gamma, \mu$ only depend on the optics and are independent of any specific particle's initial conditions.

Sec. 3.3

Visualization of Motion in Periodic Focusing System

Typical trajectory

- Slow simple harmonic oscillator-like behavior (secular motion) + Fast oscillatory motion with lattice period:



Maximum envelope a particle with arbitrary initial conditions can have

Trace space plot in periodic focusing system

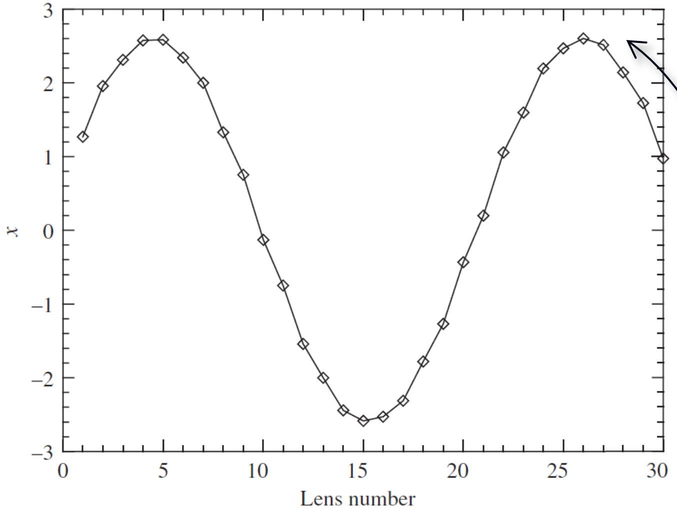


Fig. 3.5 Motion of a particle in a FODO channel with $\mu = 33°$. Lenses are at positions marked with diamond symbols. Note the deviation from simple harmonic motion occurring with the FODO period.

$$\boxed{\frac{360°}{33°} \sim 11 \text{ periods} \sim 22 \text{ lenses}}$$

→ The fast motion, despite its small spatial amplitude, will also be seen to have relatively large angles associated with it.

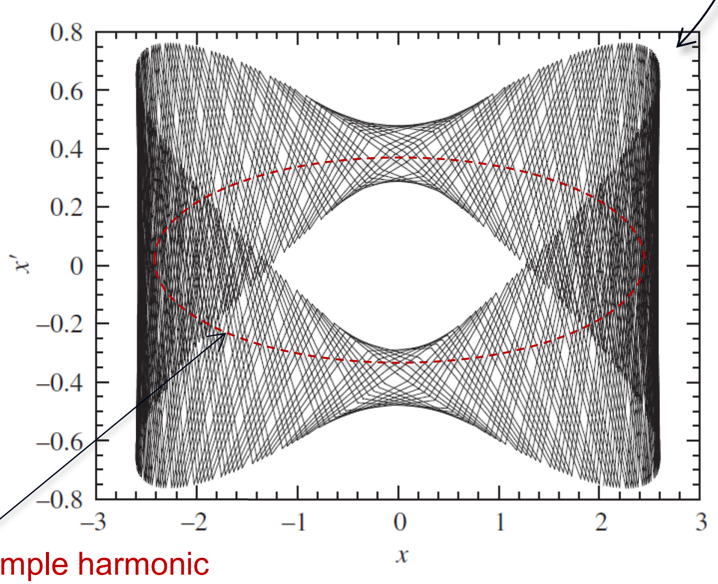


Fig. 3.6 Motion of a particle in a FODO channel of $\mu = 33°$, plotted in trace space. The fast deviations from simple harmonic motion occurring with the FODO period have a large angular spread.

→ The fast errors in the trajectory have large angular oscillations, and the trace space plot fills in a distorted annular region, yielding unclear information about the nature of the trajectory

For simple harmonic oscillator case

Poincare plot (Stroboscopic plot)

- If one only plots the trace space point of a trajectory once per FODO period, then the motion is regular.

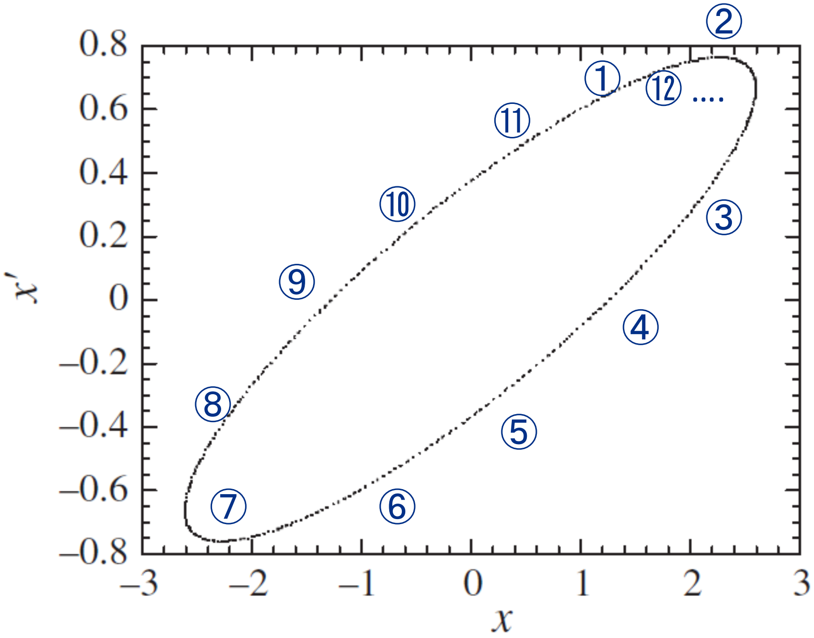


Fig. 3.7 Poincaré plot of the motion of a particle in a FODO channel of $\mu = 33°$, shown previously in Fig. 3.6, but here plotted only at the end of every FODO array.

- Note:
  - In fact, it is an ellipse in trace space.
  - However, the ellipse does not necessarily align with (x, x') axes, but it is aligned to the eigenvector axes.
  - Depending on z-position in the lattice, the Poincare plots yield different ellipses.
  - In general, particles are moving in the clockwise direction.

Smooth approximation

- We will employ here assumes that the motion can be broken down into two components, one which contains the small amplitude fast oscillatory motion (the perturbed part of the motion), and the other that contains the slowly varying or secular, large amplitude variations in the trajectory.

$$x = x_{osc} + x_{sec}$$

- Only averaging focusing effect is used in the equation of motion:

$$x'' + \kappa_x^2(z)x = 0 \quad \text{with} \quad \kappa_x^2(z) = \kappa_x^2(z + L_p) \quad \longrightarrow \quad x'' + k_{sec}^2 x = 0$$

- The averaging focusing strength can be simply deduced from

$$k_{sec} \approx \frac{\mu}{L_p}$$

[Example]

– For Thin FODO lattice:

$$k_{sec}^2 \approx \frac{1}{32}\frac{\kappa_0^4}{L_p^2}$$

– For sinusoidally varying focusing (Mathieu equation or ponderomotive force)

$$k_{sec}^2 \approx \frac{1}{8\pi^2}\frac{\kappa_0^4}{L_p^2}$$

Secs. 2.4.1/2.4.2/2.4.6 of UP-ALP

Analytic approach
for Hill’s equation

2.4.1 Pseudo-harmonic oscillations

- Let's try for the solution of the Hill's equation in the following form:

A constant determined by initial conditions of the particle

A constant determined by initial conditions of the particle

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\left[\phi(s) - \phi\right]$$

Beta function, proportional to the square of the envelope of the oscillation

Phase change of the oscillation: betatron phase

$$x'(s) = \frac{\beta'(s)}{2}\sqrt{\frac{\epsilon}{\beta(s)}}\cos[\phi(s)-\phi] - \phi'(s)\sqrt{\epsilon\beta(s)}\sin[\phi(s)-\phi]$$

$$x''(s) = \underbrace{\left[\frac{\beta''(s)}{2\sqrt{\beta(s)}} - \frac{\beta(s)'^2}{4\beta(s)^{3/2}} - \sqrt{\beta(s)}\phi'^2(s)\right]}_{= -k(s)\sqrt{\beta(s)}}\sqrt{\epsilon}\cos[\phi(s)-\phi] - \underbrace{\left[\phi''(s)\sqrt{\beta(s)} + \frac{\beta'(s)\phi'(s)}{\sqrt{\beta(s)}}\right]}_{= 0}\sqrt{\epsilon}\sin[\phi(s)-\phi]$$

- New differential equations (depending only on the magnetic lattice)

$$\frac{1}{2}\beta(s)\beta''(s) - \frac{1}{4}\beta'^2(s) + k(s)\beta^2(s) = 1 \qquad \phi'(s) = \frac{1}{\beta(s)}$$

Envelope equation

Phase advance equation

2.4.2 Principal trajectory

- By defining alpha function as $\alpha(s) = -\frac{\beta'(s)}{2}$ ←

Meaning of the alpha function:
slope of the change in the envelope
($\alpha > 0$: converging, $\alpha < 0$: diverging)

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\left[\phi(s) - \phi\right] \qquad x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\{\sin[\phi(s) - \phi] + \alpha(s)\cos[\phi(s) - \phi]\}$$

- With the following initial conditions:

$$\beta(s = s_0) = \beta_0, \quad \alpha(s = s_0) = \alpha_0, \quad \phi(s = s_0) = 0$$

$$x(s = s_0) = x_0 = \sqrt{\epsilon\beta_0}\cos\left[-\phi\right] \qquad x'(s = s_0) = x_0' = -\sqrt{\frac{\epsilon}{\beta_0}}\{\sin[-\phi] + \alpha_0\cos[-\phi]\}$$

$$\longrightarrow \sqrt{\epsilon}\cos\phi = \frac{x_0}{\sqrt{\beta_0}}, \quad \sqrt{\epsilon}\sin\phi = \alpha_0\frac{x_0}{\sqrt{\beta_0}} + \beta_0 x_0'$$

- Using trigonometric identities:

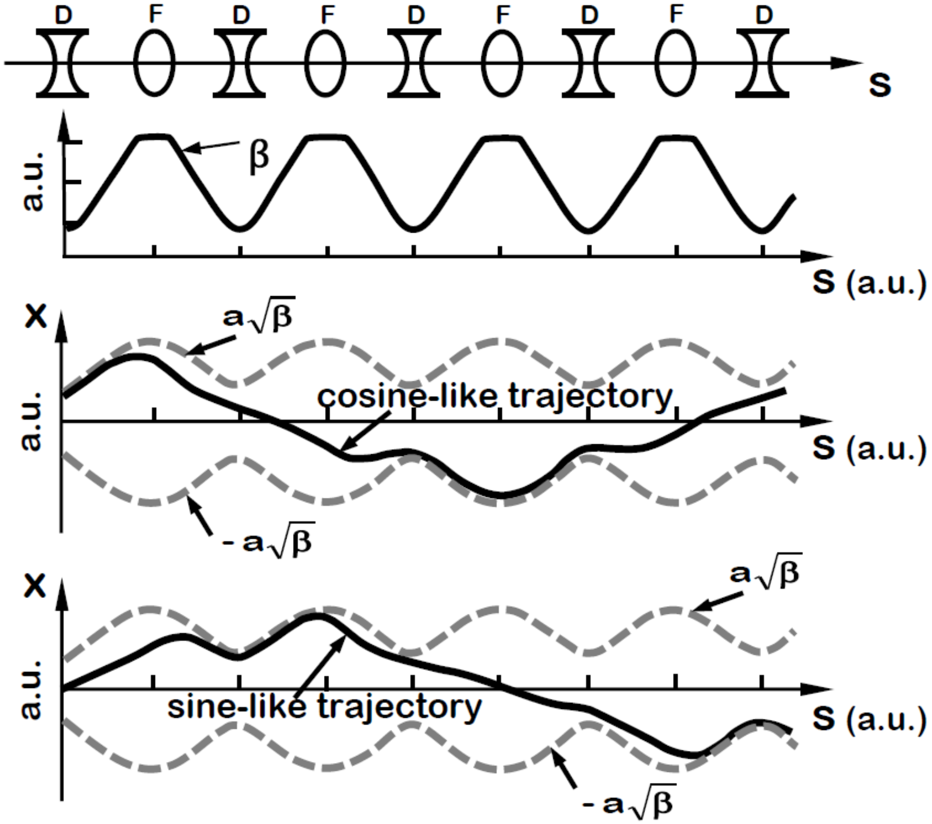
$$\begin{aligned} x(s) &= \sqrt{\epsilon\beta(s)}\cos\left[\phi(s) - \phi\right] = \sqrt{\epsilon\beta(s)}\left[\cos\phi(s)\cos\phi + \sin\phi(s)\sin\phi\right] \\ &= x_0\left[\sqrt{\frac{\beta(s)}{\beta_0}}\{\cos\phi(s) + \alpha_0\sin\phi(s)\}\right] + x_0'\left[\sqrt{\beta(s)\beta_0}\sin\phi(s)\right] \\ &\equiv x_0 C(s) + x_0' S(s) \end{aligned}$$

2.4.2 Principal trajectory (cont'd)

- Cosine-like and Sine-like solutions:

$$C(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\{\cos\phi(s) + \alpha_0 \sin\phi(s)\}, \quad C(s_0) = 1, \quad C'(s_0) = 0$$

$$S(s) = \sqrt{\beta(s)\beta_0}\sin\phi(s), \quad S(s_0) = 0, \quad S'(s_0) = 1$$



General solution is a linear combination of the cosine-like and sine-like trajectories.

2.4.7 Connection with matrix formalism

- The elements of the transfer matrix can be expressed via the Twiss functions (α, β, γ) at the beginning and end of the beam line:

$$\begin{aligned} x(s) &= x_0 C(s) + x_0' S(s) \\ x'(s) &= x_0 C'(s) + x_0' S'(s) \end{aligned}$$

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$\mathbf{M}_{s_0 \to s} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\beta(s)}{\beta_0}} \{\cos\Delta\phi + \alpha_0 \sin\Delta\phi\} & \sqrt{\beta(s)\beta_0} \sin\Delta\phi \\ -\frac{(\alpha(s)-\alpha_0)\cos\Delta\phi + (1+\alpha(s)\alpha_0)\sin\Delta\phi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\frac{\beta(s)}{\beta_0}} \{\cos\Delta\phi - \alpha(s)\sin\Delta\phi\} \end{bmatrix}$$

where

$$\Delta\phi = \phi(s) - \underbrace{\phi(s_0)}_{=0} = \phi(s) = \int_{s_0}^{s} \frac{ds'}{\beta(s')}$$

- One can also have the following decomposition:

$$\begin{aligned} \mathbf{M}_{s_0 \to s} &= \begin{bmatrix} \sqrt{\beta(s)} & 0 \\ -\frac{\alpha(s)}{\sqrt{\beta(s)}} & \frac{1}{\sqrt{\beta(s)}} \end{bmatrix} \times \begin{bmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{bmatrix} \\ &= \mathbf{B}(s) \begin{bmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{bmatrix} \mathbf{B}^{-1}(s_0) \end{aligned}$$

2.4.7 Connection with matrix formalism

- So far, we haven't yet assumed any periodicity in the transfer line. However, we may consider a periodic machine, and then the transfer matrix over a single turn (or single lattice period) would reduce to

$$\begin{aligned} \mathbf{M}_{s_0 \rightarrow s_0+L_p} &= \begin{bmatrix} \cos\Delta\phi + \alpha_0 \sin\Delta\phi & \beta_0 \sin\Delta\phi \\ -\frac{(1+\alpha_0^2)}{\beta_0} \sin\Delta\phi & \cos\mu - \alpha_0 \sin\Delta\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{bmatrix} \end{aligned}$$

When we impose periodic boundary condition on the beta function

where we define gamma function

$$\gamma_0 = \frac{1+\alpha_0^2}{\beta_0}$$

and phase advance for one turn (or one period)

$$\mu = \Delta\phi$$

2.5.1 Courant-Snyder invariant

- Hill’s equation have a remarkable property: they have an invariant!

$$x(s) = \sqrt{\epsilon\beta(s)}\cos\left[\phi(s) - \phi\right] \qquad x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\{\sin[\phi(s) - \phi] + \alpha(s)\cos[\phi(s) - \phi]\}$$

$$\longrightarrow \sqrt{\epsilon}\cos\left[\phi(s) - \phi\right] = \frac{x(s)}{\sqrt{\beta(s)}}, \quad \sqrt{\epsilon}\sin\left[\phi(s) - \phi\right] = \frac{\alpha(s)x(s)}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}x'(s)$$

- Using trigonometric identities:

$$\left(\frac{x(s)}{\sqrt{\beta(s)}}\right)^2 + \left(\frac{\alpha(s)x(s)}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}x'(s)\right)^2 = \epsilon = const.$$

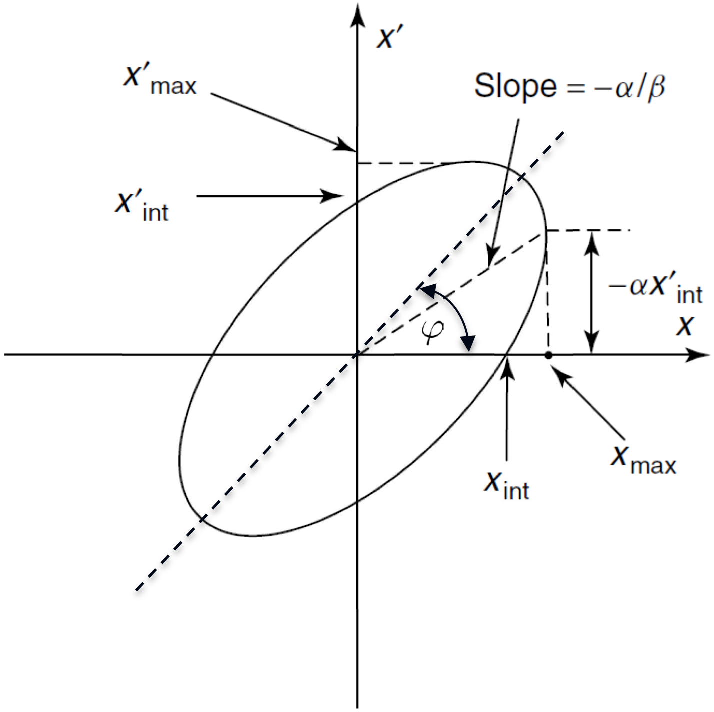
$$\epsilon = \beta(s)x'^2(s) + 2\alpha(s)x(s)x'(s) + \gamma(s)x^2(s) = \beta(s_0)x'^2(s_0) + 2\alpha(s_0)x(s_0)x'(s_0) + \gamma(s_0)x^2(s_0)$$

This invariant is known as Courant-Snyder invariant: Even though an initial point in the trace space $(x(s_0), x'(s_0),)$ changes to a different position $(x(s), x'(s),)$, the Twiss parameters (α, β, γ) change at the same time in such as way that ϵ remains constant.

2.5.1 Phase space (or trace space) ellipse

- The Courant-Snyder invariant defines an (tilted) ellipse in phase space (x, x'):

$$\epsilon = \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) = \left(\frac{x(s)}{\sqrt{\beta(s)}}\right)^2 + \left(\frac{\alpha(s)x(s)}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}x'(s)\right)^2$$



$$\tan 2\varphi = \frac{2\alpha}{\gamma - \beta}$$

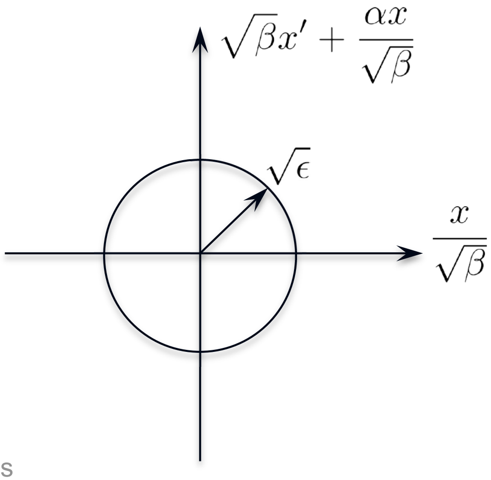
$$\text{Area in phase-space} = \pi\epsilon = const.$$

$[\epsilon]$ = m-rad, or mm-mrad, or π mm-mrad

$$x_{max} = \sqrt{\epsilon\beta}, \quad x_{int} = \sqrt{\epsilon/\gamma}$$
$$x'_{max} = \sqrt{\epsilon\gamma}, \quad x'_{int} = \sqrt{\epsilon/\beta}$$

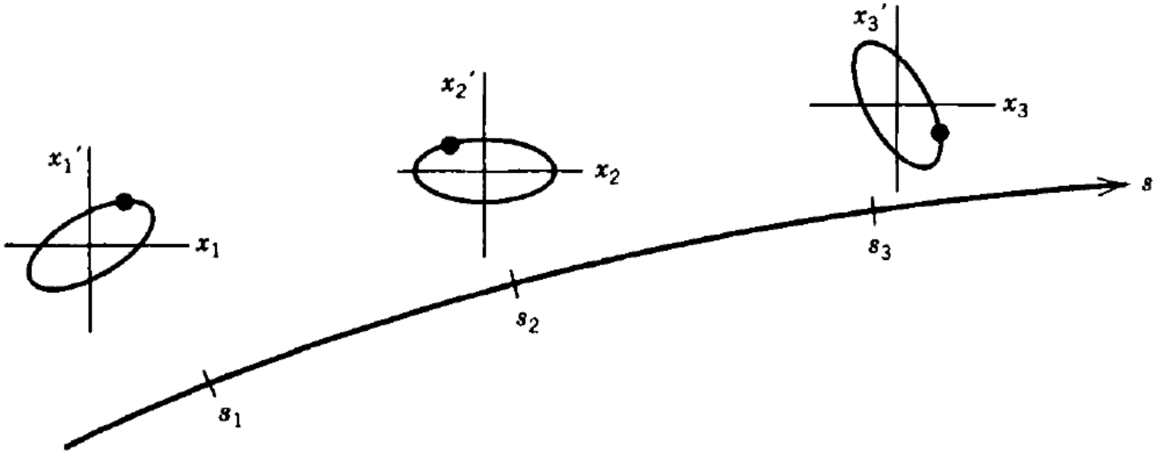
- Or, in the normalized coordinates, it defines a circle:

$$\epsilon = \left(\frac{x(s)}{\sqrt{\beta(s)}}\right)^2 + \left(\frac{\alpha(s)x(s)}{\sqrt{\beta(s)}} + \sqrt{\beta(s)}x'(s)\right)^2 = x_n^2 + x_n'^2$$

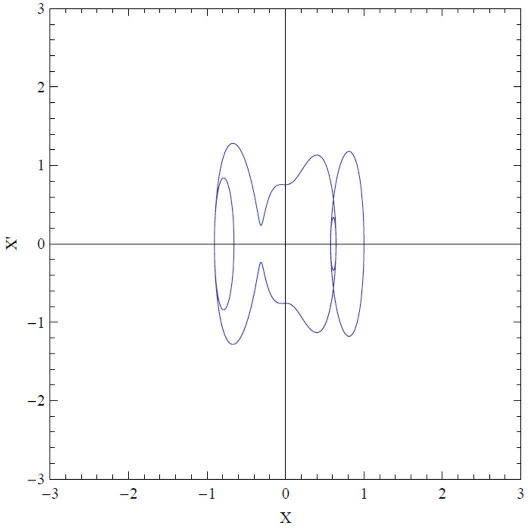


[Example]

- The shape and orientation of the ellipse keep changing as it moves along.

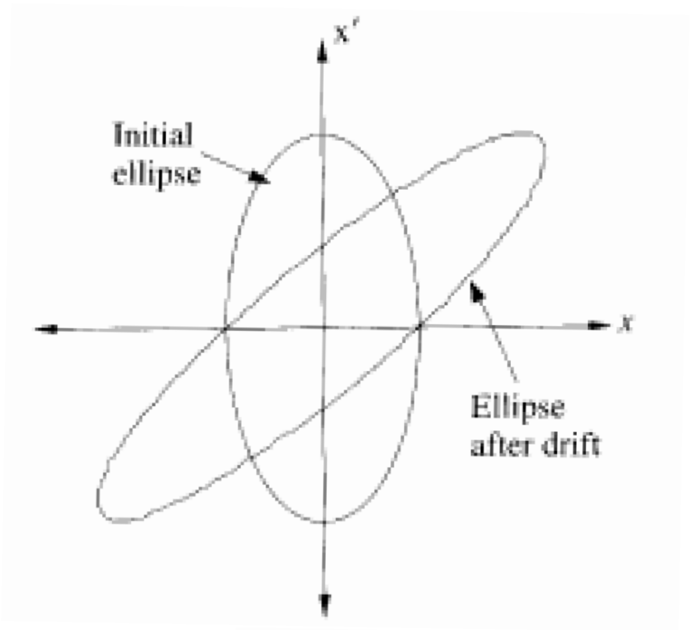


- Although the particle trajectory seems often ugly when plotted continuously, however, at a given position it will stay on some ellipse.



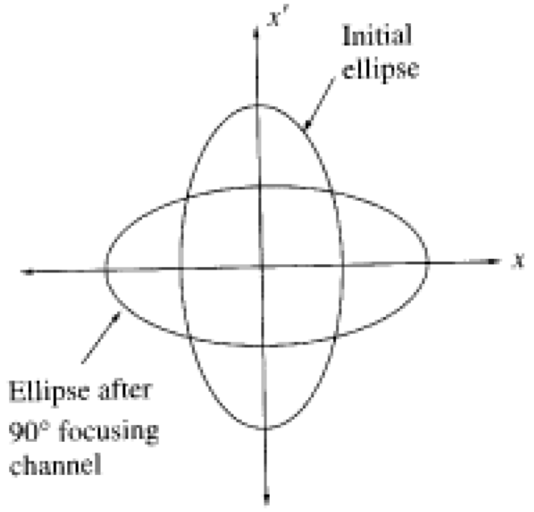
[Example]

Simple drift:

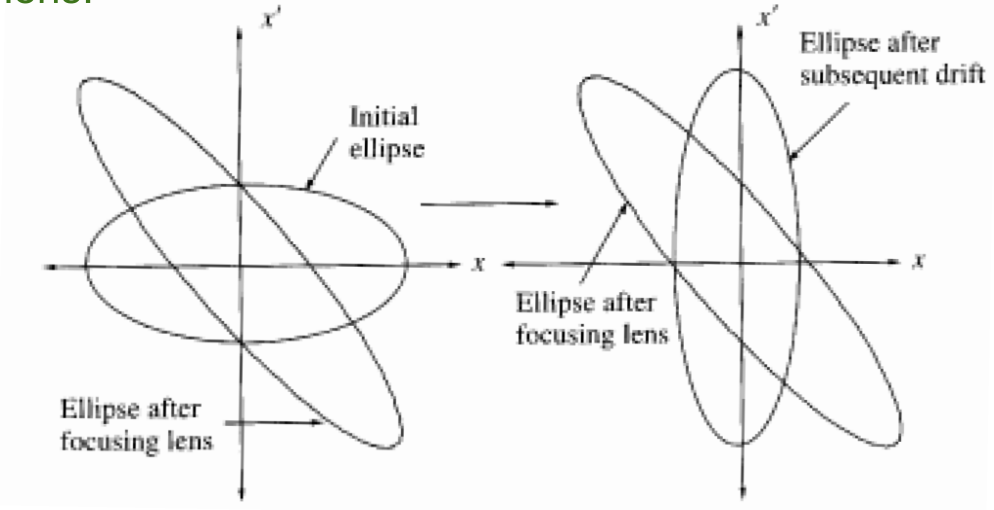


90 degree phase advance:

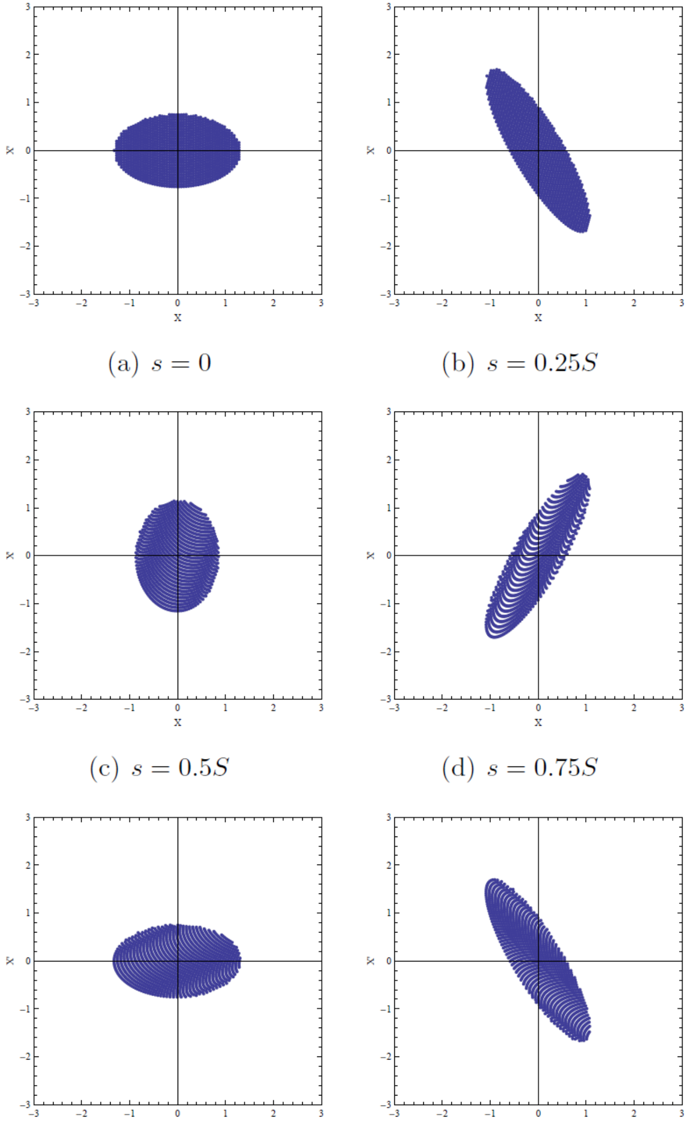
→ Minor and major axes are exchanged



Thin focusing lens:



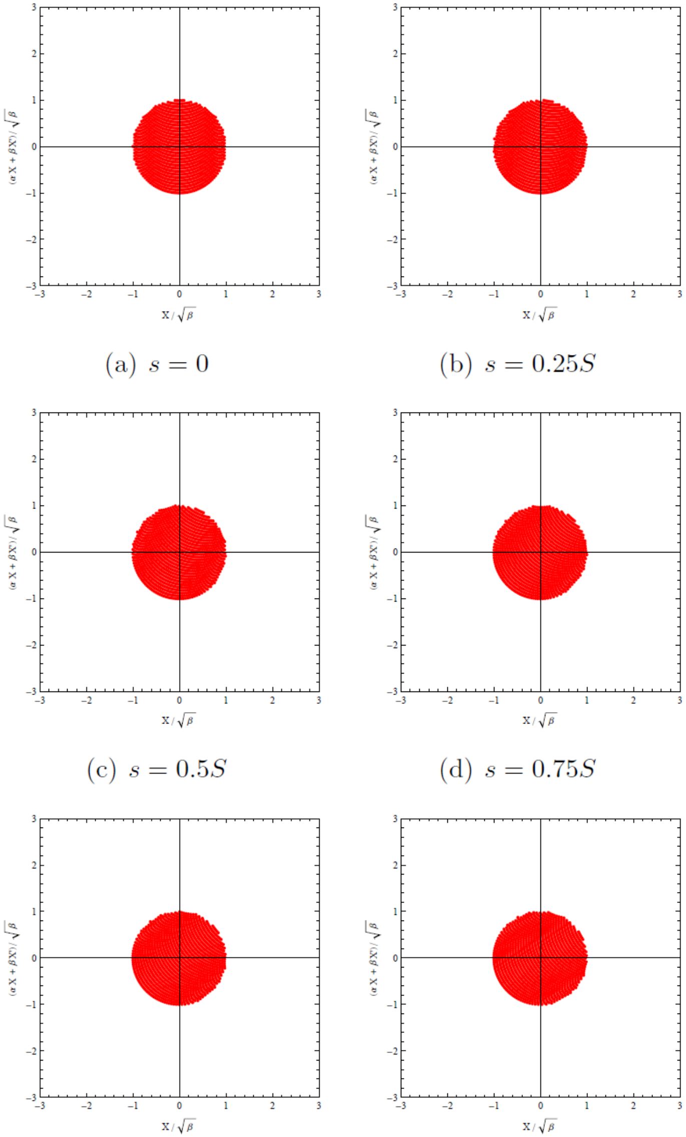
[Example] (x, x') space VS normalized coordinates



(a) $s = 0$ (b) $s = 0.25S$

(c) $s = 0.5S$ (d) $s = 0.75S$

(e) $s = 1S$ (f) $s = 1.25S$

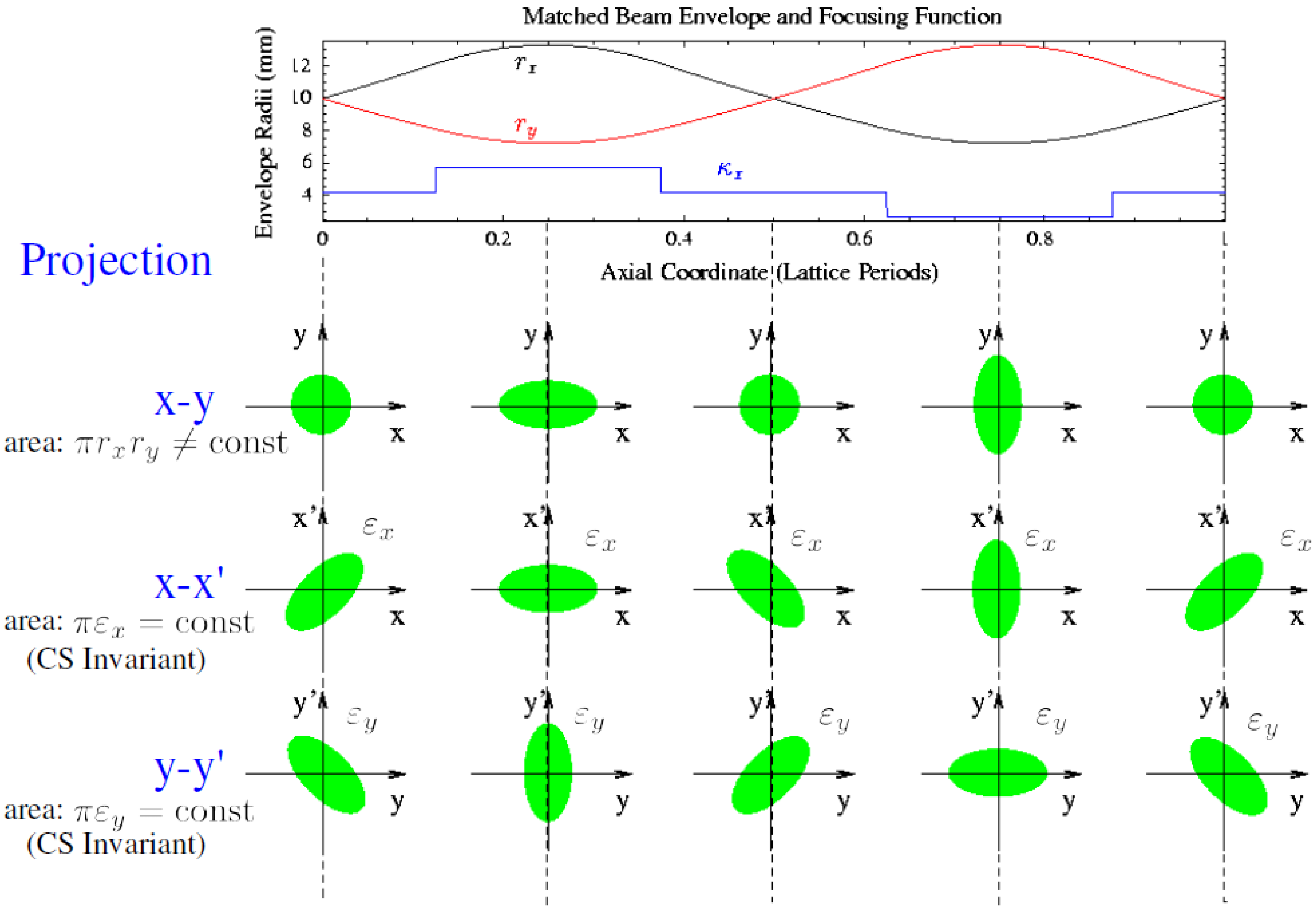


(a) $s = 0$ (b) $s = 0.25S$

(c) $s = 0.5S$ (d) $s = 0.75S$

(e) $s = 1S$ (f) $s = 1.25S$

[Example] (x, x') space VS (x, y) space



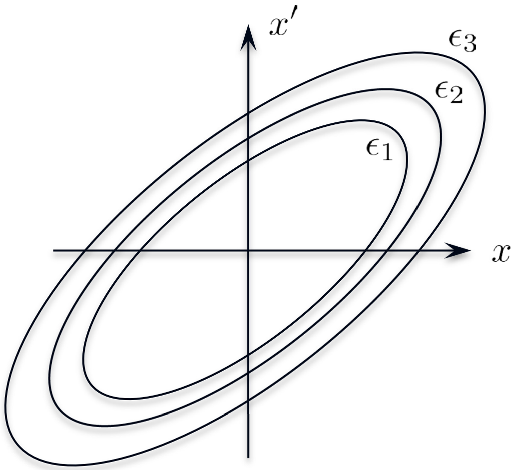
Secs. 5.2/5.3/5.4 of FOBP

Beam Distribution and Emittance

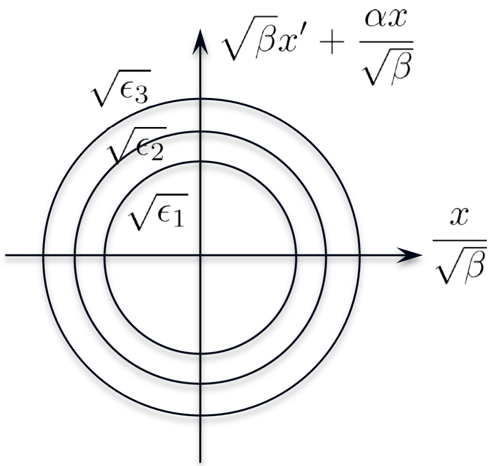
Bi-Gaussian distribution

- We assume the particle distribution is a bi-Gaussian distribution in the following form:

$$f(x,x') = \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\epsilon_{\rm rms}}\right] \propto \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] \propto \exp\left[-\frac{(x/\sqrt{\beta})^2 + (\sqrt{\beta}x' + \alpha x/\sqrt{\beta})^2}{2\epsilon_{\rm rms}}\right]$$



Constant (single particle) emittance ellipses define contours of constant phase-space distribution density



Constant (single particle) emittance circles in the normalized coordinates define contours of constant phase-space distribution density

- The rms beam emittance is proportional to the average of all the single particle emittances.
- The rms beam emittance is defined through the ellipse of the exp[-1/2] contour relative to the peak density contour.

Normalization of the distribution function

- First, check the normalization:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,x')dxdx' &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\epsilon_{\rm rms}}\right] dxdx' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{x_n^2 + x_n'^2}{2\epsilon_{\rm rms}}\right] dx_n dx_n' \\
&= \int_0^{\infty} \frac{1}{2\pi\epsilon_{\rm rms}} \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] \pi d\epsilon \\
&= 1
\end{aligned}
$$

$$
x_n = \frac{x}{\sqrt{\beta}}
$$

$$
x_n' = \sqrt{\beta}x' + \frac{\alpha x}{\sqrt{\beta}}
$$

$$
\epsilon = x_n^2 + x_n'^2
$$

- Meaning of the rms beam emittance:

Integration by parts

$$
\begin{aligned}
\langle \epsilon \rangle &= \int_0^{\infty} \epsilon \frac{1}{2\epsilon_{\rm rms}} \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] d\epsilon \\
&= \frac{1}{2\epsilon_{\rm rms}} \left\{ \epsilon(-2\epsilon_{\rm rms}) \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right]\Big|_0^{\infty} + \int_0^{\infty} 2\epsilon_{\rm rms} \exp\left[-\frac{\epsilon}{2\epsilon_{\rm rms}}\right] d\epsilon \right\} \\
&= 2\epsilon_{\rm rms}
\end{aligned}
$$

Moments of the distribution function

- From the general properties of the bi-Gaussian distribution in (x, y) plane:

https://en.wikipedia.org/wiki/Multivariate_normal_distribution

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{\delta x^2}{\sigma_x^2} - 2\rho\frac{\delta x\delta_y}{\sigma_x\sigma_y} + \frac{\delta y^2}{\sigma_y^2}\right)\right]$$

Where

$$\delta x = x - \langle x \rangle\,, \quad \delta y = y - \langle y \rangle$$

$$\sigma_x^2 = \left\langle \delta x^2 \right\rangle, \quad \sigma_y^2 = \left\langle \delta y^2 \right\rangle, \quad \sigma_{xy} = \langle \delta x \delta y \rangle \equiv \rho\sigma_x\sigma_y$$

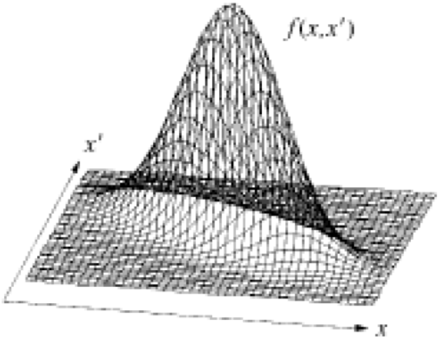
covariance

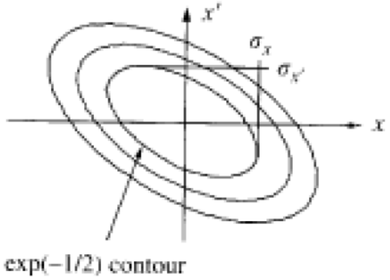
- By comparing with the beam distribution in (x, x') space:

$$\langle x \rangle = \langle x' \rangle = 0 \quad \text{when beam is alinged to its desing axis}$$

$$\sigma_x^2 = \left\langle x^2 \right\rangle = \epsilon_{\rm rms}\beta, \quad \sigma_{x'}^2 = \left\langle x'^2 \right\rangle = \epsilon_{\rm rms}\gamma, \quad \sigma_{xx'} = \langle xx' \rangle = -\epsilon_{\rm rms}\alpha$$

$$\epsilon_{\rm rms} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \rho^2\sigma_x^2\sigma_{x'}^2} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$





$\pi\epsilon_{\rm rms}$ = Area of the exp[-1/2] contour

Beam matrix

- The beam matrix is the second-order moments of the beam distribution:

Contains all the necessary information describing the beam

$$\begin{aligned}\sigma(s) = \mathbf{\Sigma}(s) &= \left\langle \mathbf{x}\mathbf{x}^T \right\rangle \\ &= \begin{bmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_x'^2 \end{bmatrix} \\ &= \underbrace{\epsilon_{\text{rms}}}_{\text{Beam property}} \underbrace{\begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}}_{\text{Lattice properties}}\end{aligned}$$

If the beam aligns with Courant-Snyder parameters

- Note that the determinant of the beam matrix is the rms emittance:

$$\det(\sigma) = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2 = \epsilon_{\text{rms}}^2$$

- If the transfer matrix is known,

$$\mathbf{x}(s) = \mathbf{M}_{s_0 \to s} \cdot \mathbf{x}(s_0)$$

$$\begin{aligned}\sigma(s) &= \left\langle \mathbf{x}(s)\mathbf{x}^T(s) \right\rangle \\ &= \left\langle \mathbf{M}_{s_0 \to s} \cdot \mathbf{x}(s_0)\mathbf{x}^T(s_0) \cdot \mathbf{M}_{s_0 \to s}^T \right\rangle \\ &= \mathbf{M}_{s_0 \to s} \cdot \sigma(s_0) \cdot \mathbf{M}_{s_0 \to s}^T\end{aligned}$$

Fraction of particles enclosed

- From the normalization of the distribution function in slide 39:

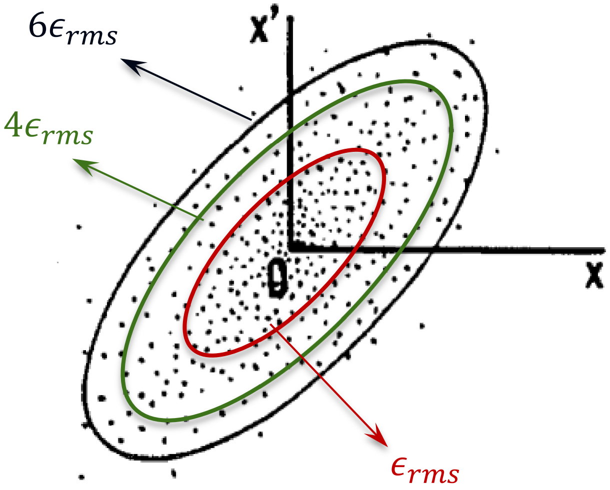
$$F = \int_0^{\epsilon_F} \frac{1}{2\epsilon_{\text{rms}}} \exp\left[-\frac{\epsilon}{2\epsilon_{\text{rms}}}\right] d\epsilon$$

  - Note that if $\epsilon_F \to \infty, F = 100\%$.
  - The ϵ_F indicates the emittance value with encloses $F(\%)$ fraction of the particles.

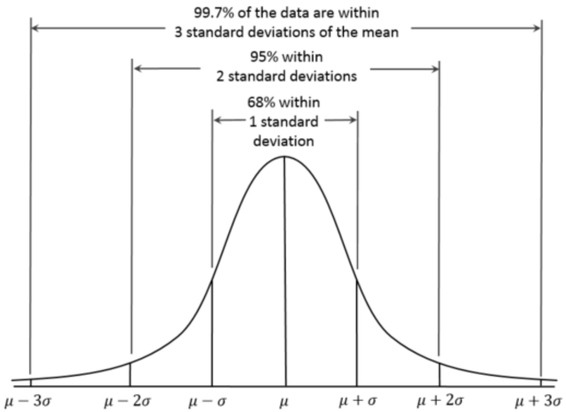
$$F = -\exp\left[-\frac{\epsilon}{2\epsilon_{\text{rms}}}\right]\Bigg|_0^{\epsilon_F} = 1 - \exp\left[-\frac{\epsilon_F}{2\epsilon_{\text{rms}}}\right]$$

$$\epsilon_F = -2\epsilon_{\text{rms}} \ln(1 - F)$$

ϵ_F	$F(\%)$
0	0
ϵ_{rms}	39%
$4\epsilon_{rms}$	87%
$6\epsilon_{rms}$	95%
∞	100%



Be careful! It is different from the single Gaussian



If the beam is not in thermal equilibrium:

- We used bi-Gaussian distribution assuming that the beam is in thermal equilibrium:

$$\frac{\partial f}{\partial t} = 0, \quad f \propto \exp\left[-\frac{H}{k_B T}\right]$$

- Even though the beam distribution function is not exactly in thermal equilibrium, it is often used as a good approximation.

- For example, in the periodic focusing system, the particle motion is always non-equillibirum, however, when plotted in trace space once per period (i.e., in the Poincare plot), we can treat the beam in equilibrium.

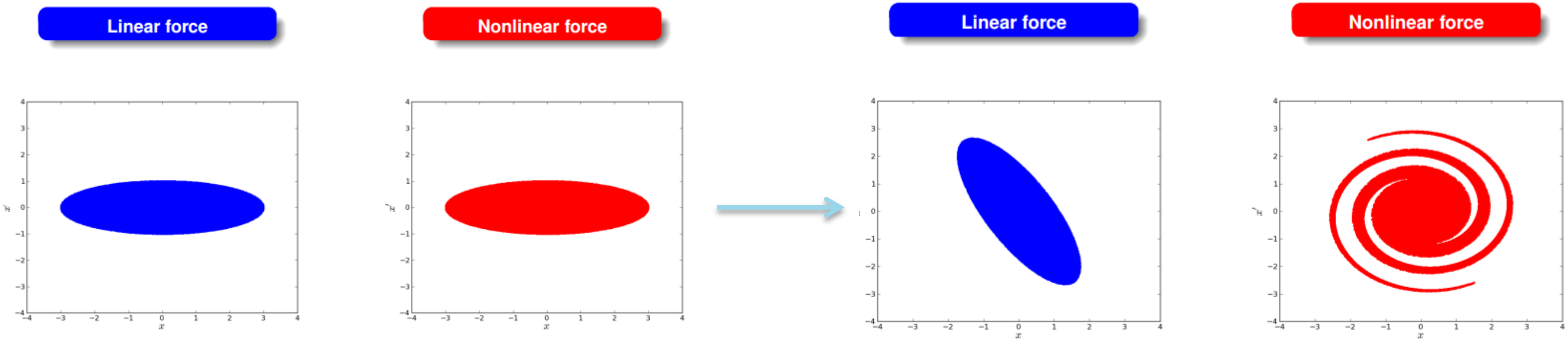
$$f(s) = f(s + L_p)$$

- Thermalization is often achieved very slowly, over many revolutions of a circular accelerator, by a combination of damping and heating effects (e.g., radiation emission, intra-beam scattering).

- In fast, transient systems, such as linear accelerators, equilibrating mechanisms (i.e. collisions) are too slow to be relevant, and if equilibria are found, they must be a property of the particle source used (Collective effects may enhance relaxation rate though).

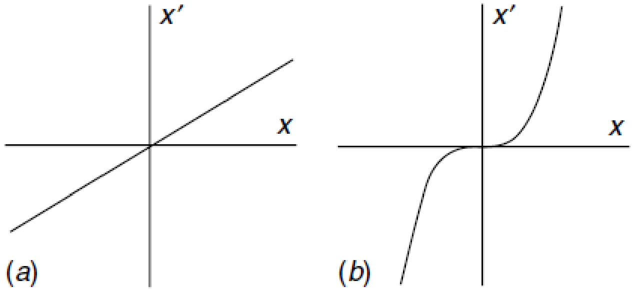
If the focusing force is not linear:

- Due to the non-linear forces, which are not included in the Courant-Snyder model, beam trajectories may not be simply ellipses.



- Non-linear forces are induced by nonlinear magnetic fields and space charge forces, and increase the rms emittance → Still we can calculate the rms emittance and 2nd moments!

- The rms emittance depends not only on the true area occupied by the beam in phase space (which is constant by Liouville theorem), but also on the distortions produced by nonlinear forces.



Phase-space area = 0
RMS emittance = 0

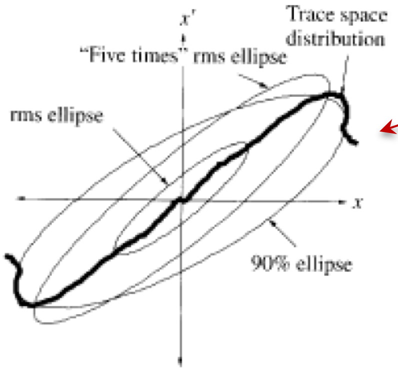
Phase-space area = 0
RMS emittance > 0

If the beam is not matching with the ellipse:

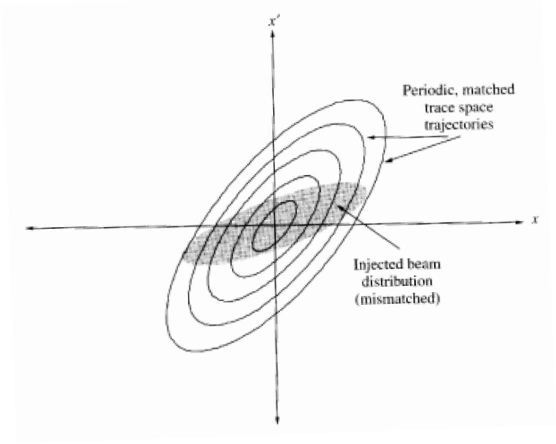
- Strictly speaking, beam’s elliptical shape and orientation determined by the second-moments may not match with the ellipse specified by the periodic lattice system:

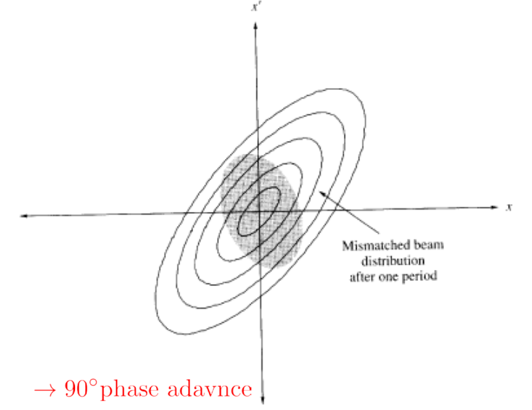
$$\beta_{beam} = \left\langle x^2 \right\rangle / \epsilon_{\text{rms}} \neq \beta_{lattice}, \quad \gamma_{beam} = \left\langle x'^2 \right\rangle / \epsilon_{\text{rms}} \neq \gamma_{lattice}, \quad \alpha_{beam} = -\left\langle xx' \right\rangle / \epsilon_{\text{rms}} \neq \alpha_{lattice}$$

- Often, even beam’s elliptical shape and orientation may not be unique. The second-moment definition of Twiss parameters can be anomalously dependent on “tail particles”.

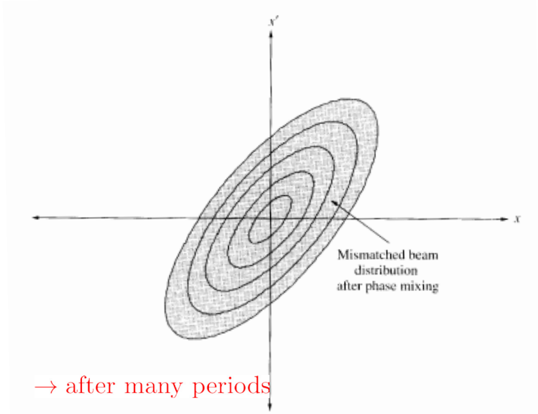


- The mismatch may seem harmless at first glance. However, amplitude-dependent tune due to small nonlinearity will eventually result in phase-mixing (or de-coherence).





→ 90°phase adavnce



→ after many periods

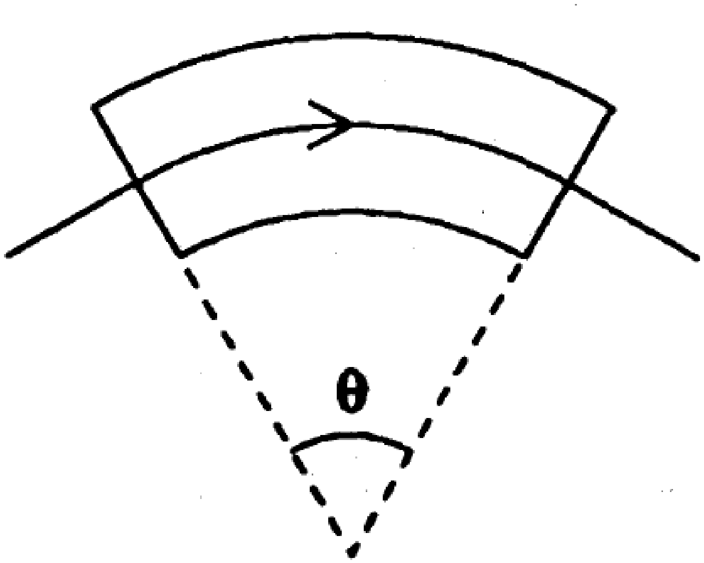
Sec. 2.4.3 of UP-ALP/ Sec. 3.5 of FOBP

Edge Focusing

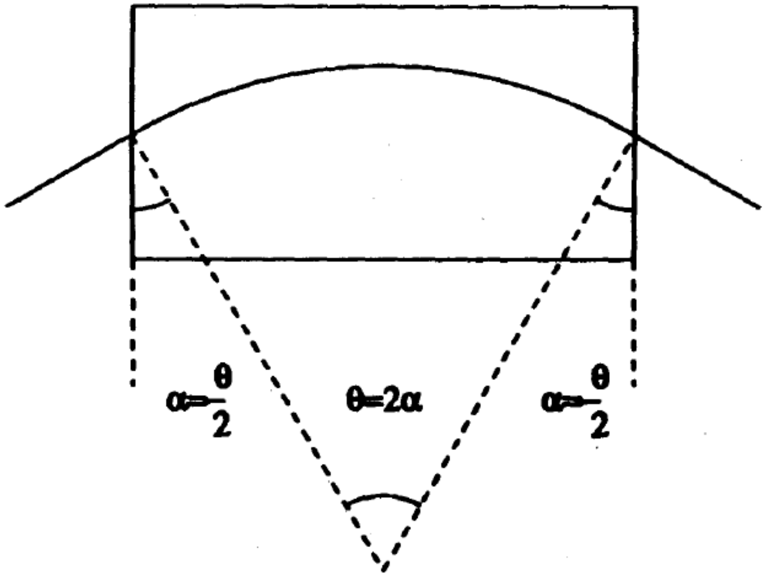
Dipoles are not infinitely long !

- Sector bend (sbend):
  - Simpler to conceptualize, but harder to build
  - Beam design entry/exit angles are ⊥ to end faces

- Rectangular bend (rbend):
  - Easier to build, but must include effects of edge focusing
  - Beam design entry/exit angles are half of bend angle



$\alpha = 0$



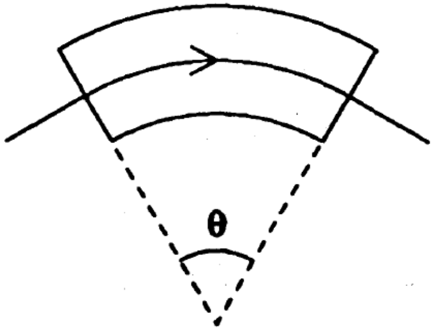
$\alpha = \frac{\theta}{2} > 0$

Transfer matrix of sbend magnet

- From Sec. 3.1 (or slide 5):

$$x'' + \left(\frac{1}{R}\right)^2 [1-n]\, x = x'' + \kappa_{b,x}^2 x = 0, \qquad y'' + \frac{n}{R^2} y = y'' + \kappa_{b,y}^2 y = 0$$

$$l = R\theta$$



- Applying the matrix formalism introduced in slide 9:

$$\begin{aligned}
\mathbf{M}_{\text{bend},x} &= \begin{bmatrix} \cos[\kappa_{b,x} l] & \frac{1}{\kappa_{b,x}} \sin[\kappa_{b,x} l] \\ -\kappa_{b,x} \sin[\kappa_{b,x} l] & \cos[\kappa_{b,x} l] \end{bmatrix} \\
&= \begin{bmatrix} \cos[\sqrt{1-n}\theta] & \frac{R}{\sqrt{1-n}} \sin[\sqrt{1-n}\theta] \\ -\frac{\sqrt{1-n}}{R} \sin[\sqrt{1-n}\theta] & \cos[\sqrt{1-n}\theta] \end{bmatrix} \\
&\xrightarrow{n=0} \begin{bmatrix} \cos[\theta] & R\sin[\theta] \\ -\frac{1}{R}\sin[\theta] & \cos[\theta] \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}_{\text{bend},y} &= \begin{bmatrix} \cos[\kappa_{b,y} l] & \frac{1}{\kappa_{b,y}} \sin[\kappa_{b,y} l] \\ -\kappa_{b,y} \sin[\kappa_{b,y} l] & \cos[\kappa_{b,y} l] \end{bmatrix} \\
&= \begin{bmatrix} \cos[\sqrt{n}\theta] & \frac{R}{\sqrt{n}} \sin[\sqrt{n}\theta] \\ -\frac{\sqrt{n}}{R} \sin[\sqrt{n}\theta] & \cos[\sqrt{n}\theta] \end{bmatrix} \\
&\xrightarrow{n=0} \begin{bmatrix} 1 & R\theta \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Simple drift in the vertical direction
if the magnet is not a combined-function magnet

Edge focusing in the vertical direction

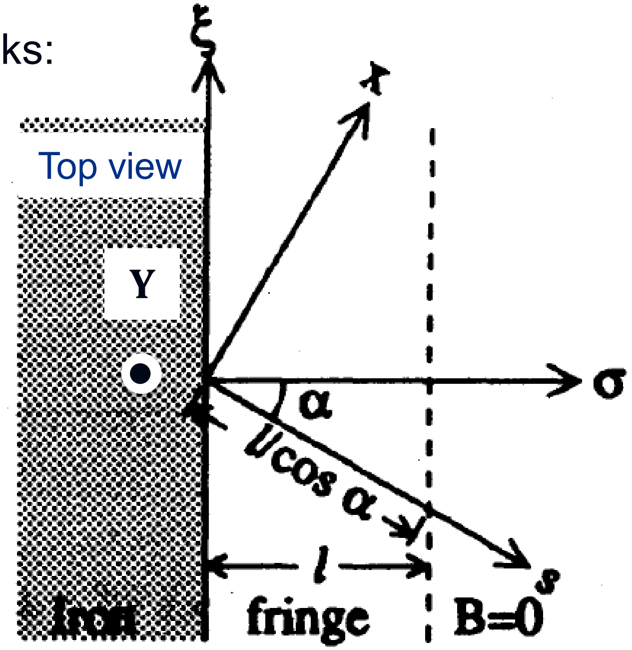
- There is a finite transverse field which induces vertical kicks:

$$B_y \approx B_0\left(1-\frac{\sigma}{l}\right) \quad \text{for} \quad 0<\sigma<l$$

$$B_\xi \approx 0 \quad \text{(i.e., assuming very wide poles)}$$

$$B_\sigma \simeq \cancel{B_\sigma(y=0)} + \left(\frac{\partial B_\sigma}{\partial y}\right) y \overset{\substack{\nabla\times\mathbf{B}=0 \\ \downarrow}}{=} \left(\frac{\partial B_y}{\partial \sigma}\right) y = -\frac{B_0}{l} y$$

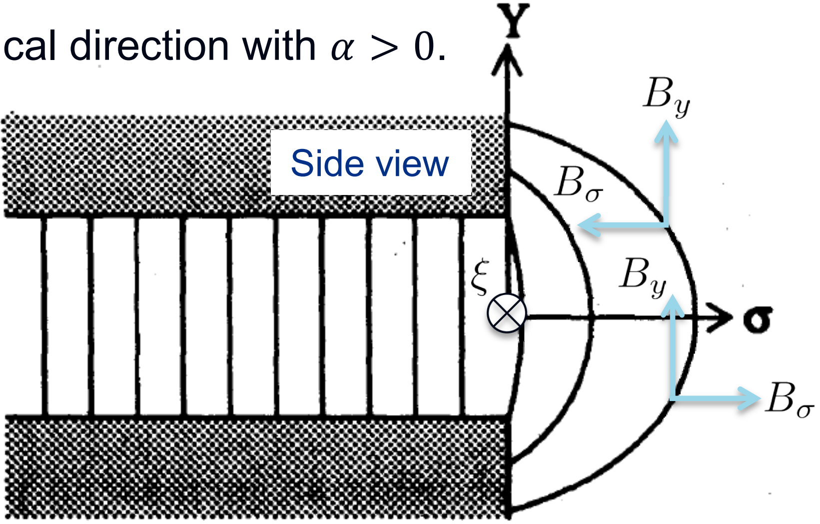
$$B_x = B_\xi \cos\alpha + B_\sigma \sin\alpha = -\frac{B_0 \sin\alpha}{l} y$$



- Focusing effect of a fringe field in the vertical direction with $\alpha > 0$.

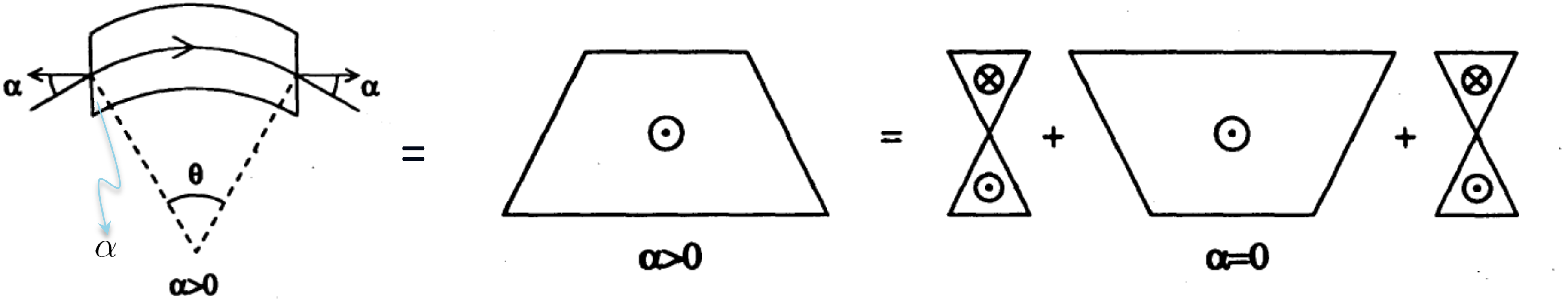
$$B_x = -\frac{B_0 \sin\alpha}{l} y$$

→ Quadrupole-like field



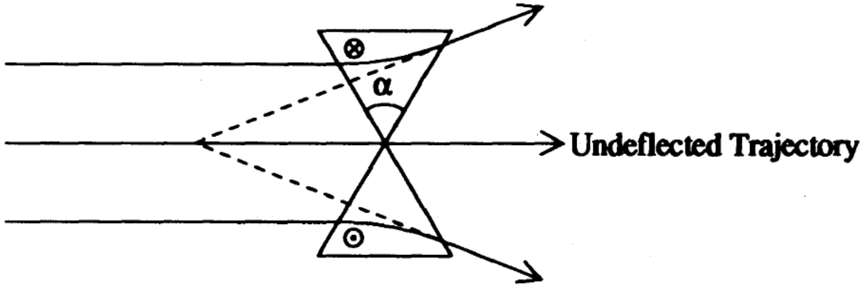
Edge (de)focusing in the horizontal direction

- For $\alpha \neq 0$, we need to include edge (de)focusing effects.



- Defocusing effect of a thin wedge in horizontal direction with $\alpha > 0$.

Top view



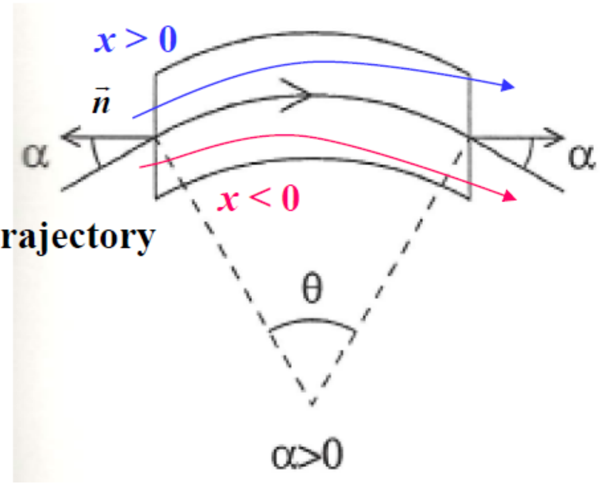
Another view of the edge focusing

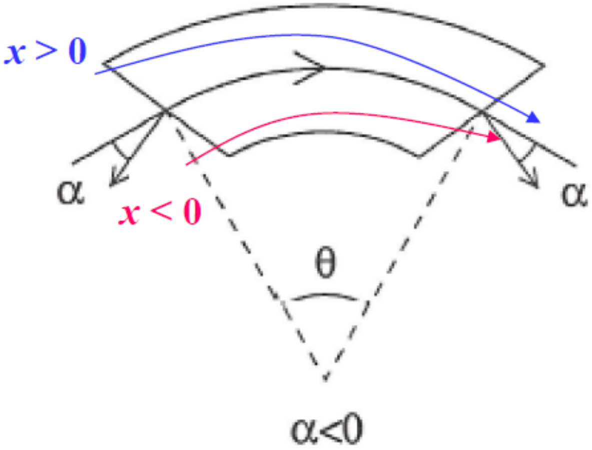
- For $\alpha > 0$,
  - Particles located at positive x take shorter paths in the dipole & to be bent weakly
  - Particles located at negative x take longer paths in the dipole & to be bent strongly

    → horizontal defocusing & vertical focusing

- For $\alpha < 0$,
  - Particles located at positive x take longer paths in the dipole & to be bent strongly
  - Particles located at negative x take shorter paths in the dipole & to be bent weakly

    → horizontal focusing & vertical defocusing





[From Dr. Yujong Kim’s KoPAS 2015 Slide]