# Lecture 2

# ‘Single’ Charged Particle Motion in Static Fields

**(Ch. 2 of FOBP, Ch. 2 of UP-ALP)**

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# 2.1 Motion in a uniform magnetic field

- Motion of charged particle in EM fields:

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mathbf{F}, \quad \frac{d\mathcal{E}}{dt} = \mathbf{F} \cdot \mathbf{v}$$

  - where the momentum $\boldsymbol{p}$ and energy $\mathcal{E}$ of the particle are given by $\mathbf{p} = m_0 \gamma \mathbf{v}, \quad \mathcal{E} = m_0 \gamma c^2$
  - and $m_0$ is particle rest mass, and $\gamma$ is the relativistic mass factor.

- In a uniform magnetic field: Conservation of energy

$$m_0 \frac{d\gamma \mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}, \quad \frac{d\mathcal{E}}{dt} = 0 \longrightarrow \mathcal{E} = const., \quad p = \sqrt{p_\perp^2 + p_z^2} = const.$$

- Equations of motion:

$$m_0 \gamma \dot{v}_x = q v_y B \ \text{ and } \ m_0 \gamma \dot{v}_y = -q v_x B, \quad \dot{v}_z = 0$$

$$\ddot{v}_x = \frac{qB}{m_0 \gamma} \dot{v}_y = -\left(\frac{qB}{m_0 \gamma}\right)^2 v_x$$

- Cyclotron frequency:

$$\omega_c = \frac{qB}{\gamma m_0}$$

# 2.1 Motion in a uniform magnetic field (cont'd)

- General solutions:

$$v_x = -v_m \sin(\omega_c t + \phi), \quad v_y = v_m \cos(\omega_c t + \phi)$$

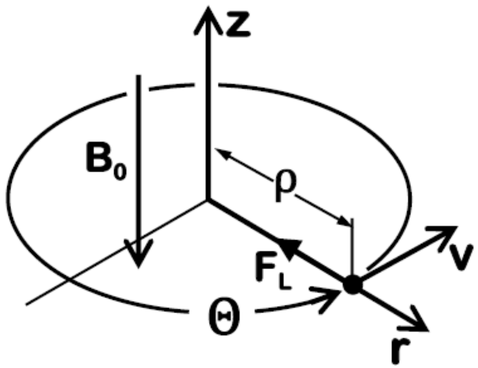
$$x = R\cos(\omega_c t + \phi) + x_0, \quad y = R\sin(\omega_c t + \phi) + y_0$$

- Meaning of the parameters:

$$v_x^2 + v_y^2 = v_m^2 = v_\perp^2, \quad R = v_m/\omega_c = \rho, \quad (x - x_0)^2 + (y - y_0)^2 = R^2$$

- Physical meaning: Balance of radial force (Lorentz force, $F_L$) and centripetal force

$$\frac{\gamma m_0 v_\perp^2}{R} = q v_\perp B_0 \quad \text{or} \quad p_\perp = q B_0 R$$




For q >0

- Magnetic rigidity [T m]:

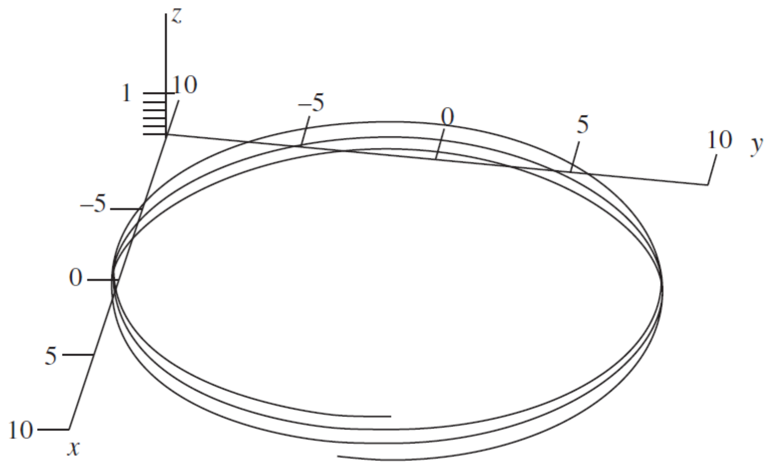
$$B_0 R = \frac{p_\perp}{q}, \quad \text{or} \quad B\rho = \frac{p}{q}$$

- Pitch angle: Associated with helical motion

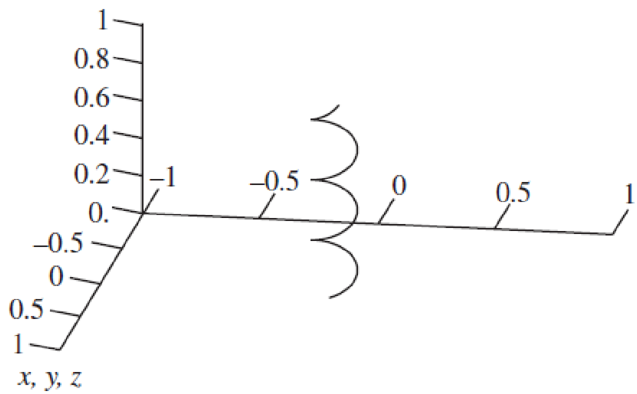
$$\theta_p = \tan^{-1}(p_z/p_\perp)$$

# [Note] Circular Accelerator VS Focusing Solenoid

- Circular accelerator:
  - Very small pitch angle
  - Indeed, $p_z \approx 0$
  - Circular design orbit




- Focusing solenoid:
  - Very large pitch angle
  - Indeed, $p_z \gg p_\perp$
  - Straight design orbit

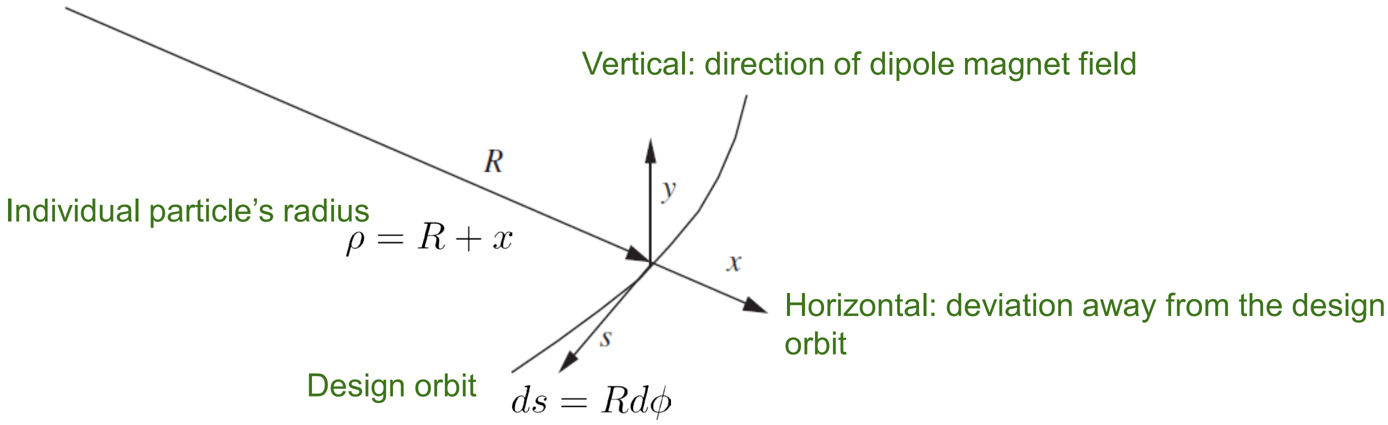



→Be careful ! Later we will reassign x, y, and z coordinates.
→In fact, B fields are applied locally around the particle orbits.

# 2.2 Circular accelerator

- We analyze the charged particle dynamics near the design orbit. The design orbit is specified by a certain radius of curvature ($R$) and a certain momentum ($p_0 = qB_0R$)

- A new locally defined right-handed coordinate system:




- Equation of motion in this new coordinate system: Homework (Problem 2.1 of FOBP)

$$\frac{dp_\rho}{dt} = \frac{\gamma m_0 v_\phi^2}{\rho} - q v_\phi B_0 \qquad (2.9)$$

- The azimuthal velocity and radial momentum:

Reference

$$v_\phi = \rho\dot{\phi} \neq \dot{s} \equiv v_0, \quad \text{and} \quad p_\rho = \gamma m_0 \dot{\rho} = \gamma m_0 \dot{x} = p_x$$

$$v_\phi \approx v_0$$

Individual particle's velocity

# 2.2 Circular accelerator (cont'd)

- Linearization of Eq. (2.9) by assuming $x \ll R$: Lowest order Taylor series expansion about the design orbit equilibrium ($p_x = p_\rho = 0$ at $\rho = R$),

$$\frac{dp_x}{dt} = \frac{\gamma m_0 v_0^2}{R(1+x/R)} - qv_0 B_0 \simeq \frac{\gamma m_0 v_0^2}{R}(1 - x/R + \cdots) - qv_0 B_0$$

$$\simeq -\frac{\gamma m_0 v_0^2}{R^2}x + \cancel{\frac{\gamma m_0 v_0^2}{R}} - \cancel{qv_0 B_0}$$

- Using the definition of the design radius and cyclotron frequency:

$$\frac{d^2x}{dt^2} + \omega_c^2 = 0$$

- Using $s$ as the independent variable:

$$(\cdots)' \equiv \frac{d}{ds} = \frac{d}{v_0 dt}$$

$$x'' + \left(\frac{1}{R}\right)^2 x = 0$$

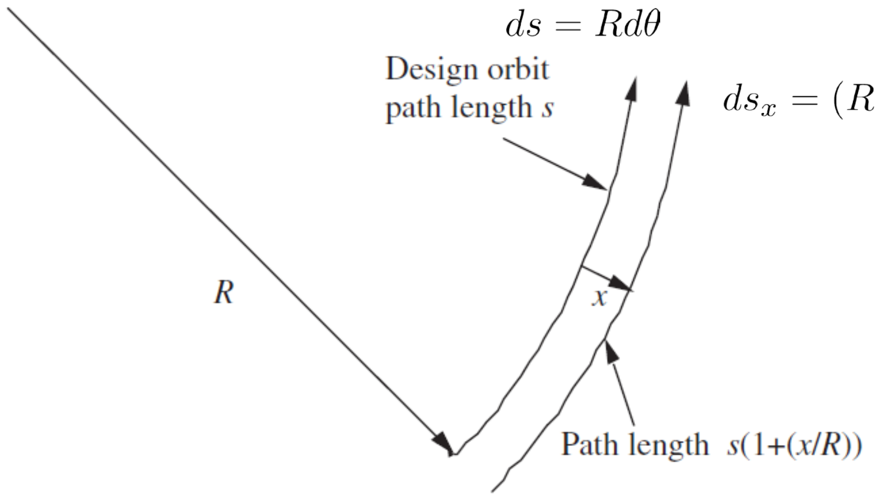
- Simple harmonic oscillations about the design orbit for particles having the same momentum ($p_0 = qB_0R$): We call it Betatron oscillations → Basis of phase advance and tunes.

# [Note] How does the restoring force arise?

- Total momentum transfer of the particle on an arbitrary offset orbit:

$$ds = R d\theta$$

$$ds_x = (R + x)d\theta = R(1 + x/R)d\theta = (1 + x/R)ds$$




Same momentum but different path

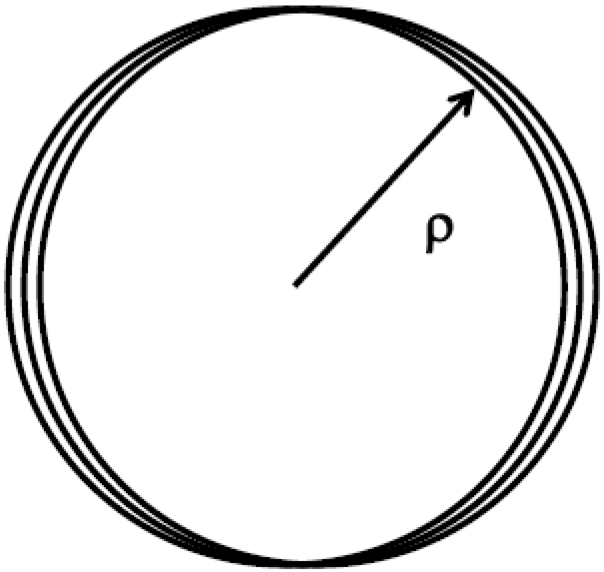
$$\Delta p_x = -q \int_{t_1}^{t_2} v_0 B_0 dt = -q \int_{s_1}^{s_2} B_0 \left(1 + \frac{x}{R}\right) ds = -q \int_{s_1}^{s_2} B_0 ds - q \int_{s_1}^{s_2} B_0 \frac{x}{R} ds$$

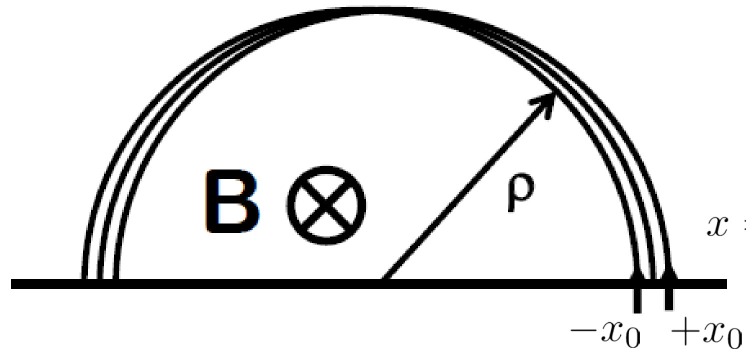
- Momentum transfer is larger (smaller) due to longer (shorter) path length: Path length focusing (basis of weak focusing system)

- Betatron oscillation is due to trajectory errors (angle and offest deviation from the design orbit) for particles that have the design momentum

# [Note] How does the restoring force arise?

- Or, we can explain in terms of an error in the center of curvature of the orbit:




$$x = x_0 \cos\left(2\pi \frac{s}{\lambda}\right)$$

$$\lambda = 2\pi\rho$$

- The trajectories of the particles in this dipole exhibit an equivalent “focusing” with the wavelength of motion (along the curvilinear coordinate $s$) given by $2\pi\rho$.

$$\frac{d^2x}{ds^2} + \frac{x}{\rho^2} = 0$$

# [Note] Momentum dispersion

- Displacement of an arbitrary particle from the design orbit due to deviation from the design momentum:

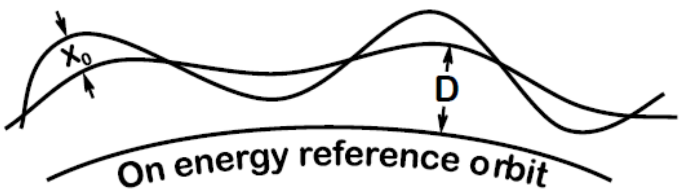
$$\delta p \equiv p - p_0$$

- An analysis which treats the particle dynamics only in a first-order Taylor series in both betatron (i.e., angle and offset) errors and momentum errors is by assumption a description which is additive in these quantities:

$$x = x_\beta + \eta_x \frac{\delta p}{p_0}$$

Betatron orbit  Dispersion

Fractional momentum deviation $= \delta_p$



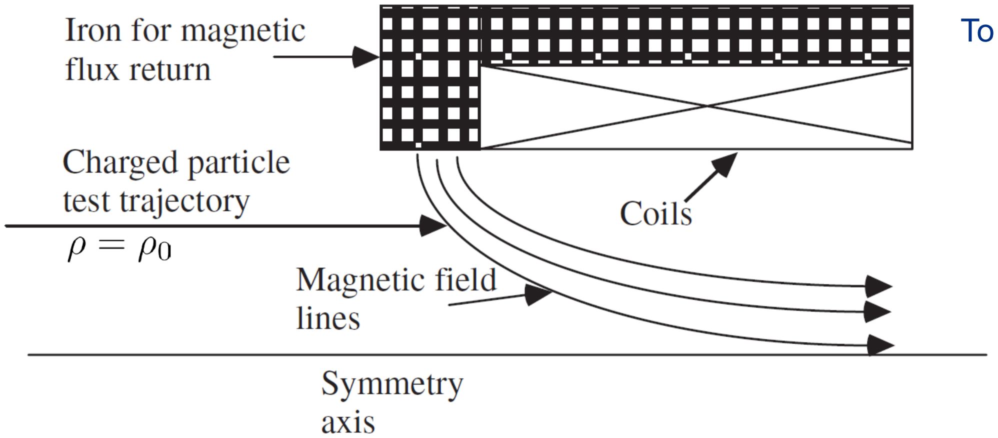

- The coefficient $\eta_x$ (or $D_x$) is termed the momentum dispersion, and is generally a detailed function of the magnetic field profile with variation in $s$.
- In case of a uniform magnetic field, the dispersion is constant at the design momentum:

$$\eta_x = \frac{\partial x}{\partial \delta_p} = \frac{\partial x}{\partial \left[\frac{\delta p}{p_0}\right]} = p_0 \frac{\partial x}{\partial (p - p_0)} = p_0 \frac{\partial R}{\partial p} = R(p_0)$$

$$R(p) = p/qB_0$$ R is a function of p

# 2.3 Focusing in solenoids

- The key to understanding the motion of a charged particle in a focusing solenoid is to recognize how the angular momentum, which drives this helical motion, arises.
- We need to ask what happens when the charged particle moves from a region where the magnetic field vanishes to one where it is uniform.




First order Taylor expansion

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \\ &= 2\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} \\ &= 0 \\ \longrightarrow \frac{\partial B_x}{\partial x} &= -\frac{1}{2}\frac{\partial B_z}{\partial z} = -\frac{1}{2}B_z' \end{aligned}$$

$$\mathbf{B} = -\frac{1}{2}B_z'(z)\left(x\hat{x} + y\hat{y}\right) + B_z(z)\hat{z} = -\frac{1}{2}B_z'(z)\rho\hat{\rho} + B_z(z)\hat{z}$$

$$B_z'(z) \equiv \left(\frac{\partial B_z}{\partial z}\right)_{(0,0)}$$

$$\mathbf{A} = \frac{1}{2}B_z(z)\left(x\hat{y} - y\hat{x}\right) = \frac{1}{2}B_z(z)\rho\hat{\theta}$$

# 2.3 Focusing in solenoids (cont'd)

- Transverse momentum kick imparted to a charged particle as it passes through the fringe field region:

Kick is $-\hat{\theta}$ direction
Rotation is CW for $q > 0$

$$\Delta p_{\perp} \simeq q \int_{t_1}^{t_2} v_z B_\rho dt = q \int_{z_1}^{z_2} B_\rho dz = -q\frac{\rho_0}{2} \int_{z_1}^{z_2} \left(\frac{\partial B_z}{\partial z}\right)_{\rho=0} dz = -q\frac{\rho_0}{2}\left[\underbrace{\cancel{B_z}(z_2)}_{B_0} - \underbrace{\cancel{B_z}(z_1)}_{0}\right] = -q\frac{\rho_0}{2}B_0$$

- Rotation in the axisymmetric system due to conservation of canonical angular momentum or Busch's theorem:

$$p_\theta = \gamma m \rho^2 \dot{\theta} + q\rho A_\theta = const.$$

$$\gamma m \rho^2 \dot{\theta} + \frac{q}{2\pi}\Phi = const.$$

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \int \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l} = 2\pi \rho A_\theta$$

$\Phi = 0$ outside the solenoid, but $\Phi \neq 0$ inside.

- A charged particle with no initial transverse motion displays helical motion inside of the solenoid, with radius of curvature such that the particle orbit passes through the axis.

$$R = \frac{|\Delta p_\perp|}{qB_0} = \frac{\rho_0}{2}$$

$$const. = \gamma m \rho_0^2 \cancel{\dot{\theta}} + \frac{q}{2\pi}\cancel{\Phi} = 0 = \gamma m \rho^2 \dot{\theta} + q\rho A_\theta \longrightarrow \rho = 0$$

# [Note] Equations of motion

- Equations of motion for constant $\gamma$:

$$
\begin{aligned}
\mathbf{F} &= q\mathbf{v}\times\mathbf{B} \\
&= q(\dot{x}\hat{x}+\dot{y}\hat{y}+\dot{z}\hat{z})\times\left[-\frac{1}{2}B_z'(x\hat{x}+y\hat{y})+B_z\hat{z}\right] \\
&= qB_z(\dot{y}\hat{x}-\dot{x}\hat{y})+\frac{q\dot{z}B_z'}{2}(y\hat{x}-x\hat{y}) \\
&= \underbrace{qB_z(\dot{y}\hat{x}-\dot{x}\hat{y})}_{\text{centripetal force}}+\frac{q\dot{B}_z}{2}(y\hat{x}-x\hat{y})
\end{aligned}
$$

$$\dot{z}\left(\frac{\partial B_z}{\partial z}\right)_{(0,0)}\approx\frac{dB_z}{dt}$$

$$
\begin{aligned}
\gamma m_0\frac{d^2x}{dt^2}-qB_z\frac{dy}{dt}-\frac{q}{2}\frac{dB_z}{dt}y &= 0 \\
\gamma m_0\frac{d^2y}{dt^2}+qB_z\frac{dx}{dt}+\frac{q}{2}\frac{dB_z}{dt}x &= 0
\end{aligned}
$$

- Introducing (normalized) Larmor frequency and applying paraxial approximation:

$$\omega_L=\frac{\omega_c}{2}=\frac{qB_0}{2\gamma m_0}=-\frac{d\theta_L}{dt}$$

Different sign convention from the FOBP
Rotation is CW for $q>0$

$$
\begin{aligned}
\frac{d^2x}{dt^2}-2\omega_L\frac{dy}{dt}-\frac{d\omega_L}{dt}y &= 0 \\
\frac{d^2y}{dt^2}+2\omega_L\frac{dx}{dt}+\frac{d\omega_L}{dt}x &= 0
\end{aligned}
$$

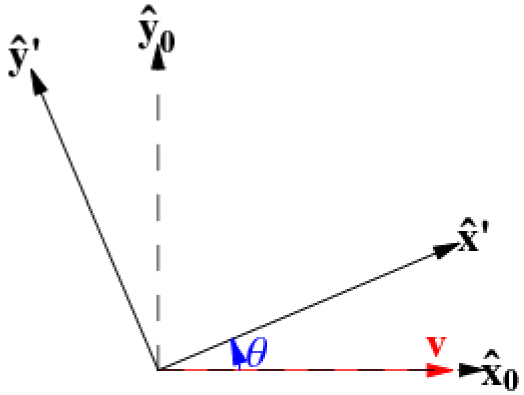
$$\Omega_L=k_L=\frac{\omega_L}{v_z}=\frac{qB_0}{2p_z}\simeq\frac{qB_0}{2p}$$

$$
\begin{aligned}
\frac{d^2x}{dz^2}-2k_L\frac{dy}{dz}-\frac{dk_L}{dz}y &= 0 \\
\frac{d^2y}{dz^2}+2k_L\frac{dx}{dz}+\frac{dk_L}{dz}x &= 0
\end{aligned}
$$

# [Note] Larmor frame

- Introducing Larmor frame (the frame that rotates about $z$-axis with normalized Larmor frequency), in which the transverse orbits in the rotating frame are related to the orbits in the laboratory frame by

$$\begin{pmatrix} x_L \\ y_L \end{pmatrix} = \begin{pmatrix} \cos\theta_L & \sin\theta_L \\ -\sin\theta_L & \cos\theta_L \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$




- By direct substitutions, one can show that

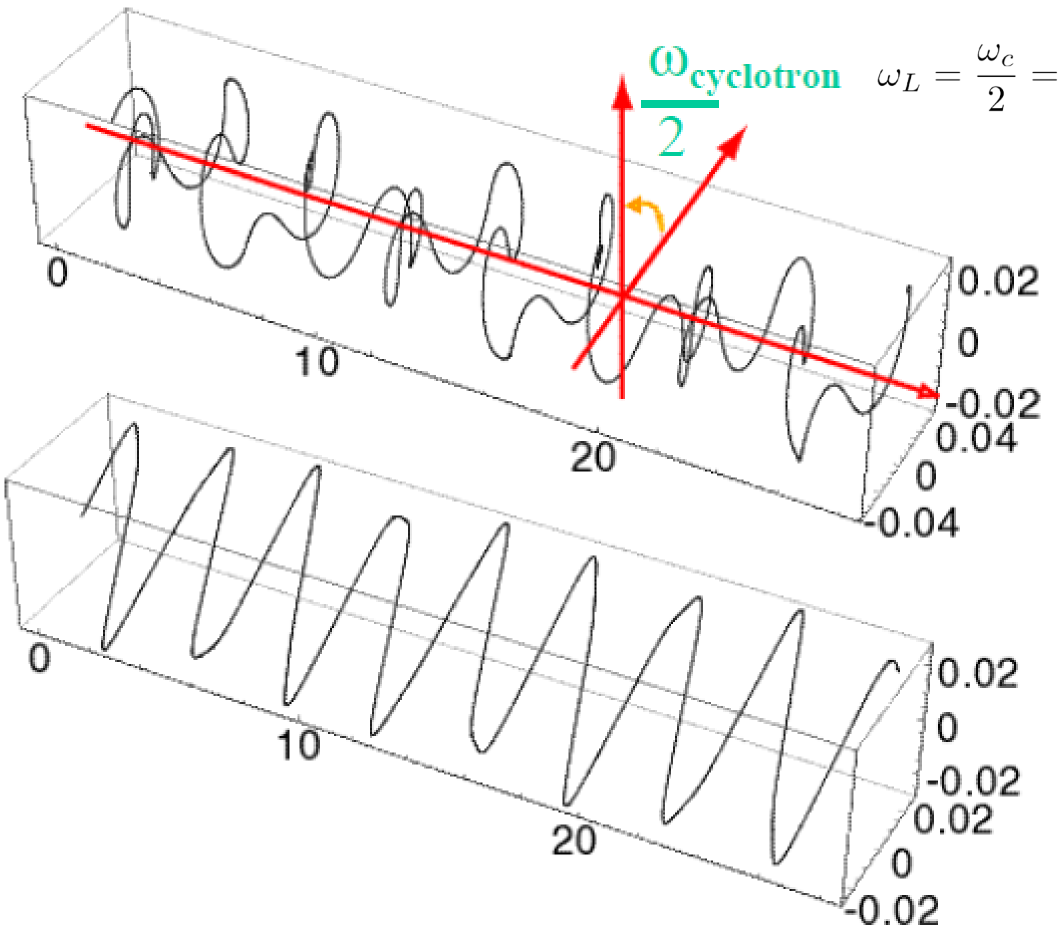
$$\begin{aligned} \frac{d^2x}{dt^2} - 2\omega_L\frac{dy}{dt} - \frac{d\omega_L}{dt}y &= 0 \\ \frac{d^2y}{dt^2} + 2\omega_L\frac{dx}{dt} + \frac{d\omega_L}{dt}x &= 0 \end{aligned} \longrightarrow \ddot{x}_L + \omega_L^2 x_L = 0, \quad \ddot{y}_L + \omega_L^2 y_L = 0$$

Uncoupled simple harmonic oscillators in the Larmor frame

$$\begin{aligned} \frac{d^2x}{dz^2} - 2k_L\frac{dy}{dz} - \frac{dk_L}{dz}y &= 0 \\ \frac{d^2y}{dz^2} + 2k_L\frac{dx}{dz} + \frac{dk_L}{dz}x &= 0 \end{aligned} \longrightarrow x_L'' + k_L^2 x_L = 0, \quad y_L'' + k_L^2 y_L = 0$$

# [Example] Larmor frame

- Particle orbits become considerably simpler by introducing Larmor frame:




$$\omega_L = \frac{\omega_c}{2} = \frac{qB_0}{2\gamma m_0} = -\frac{d\theta_L}{dt}$$

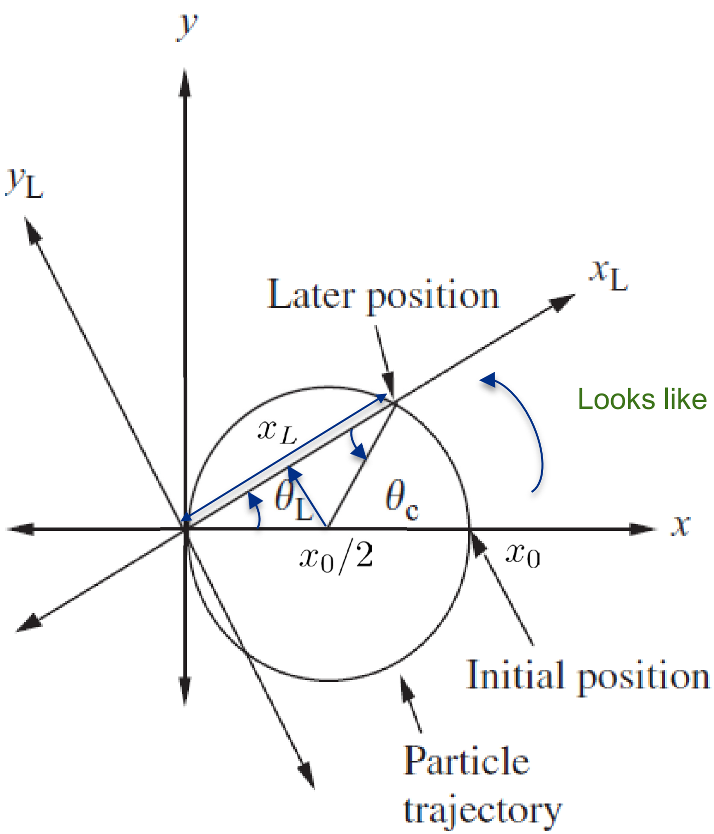
In lab frame

This is the case with some initial velocities in entering the solenoid

In Lamor frame

# [Example] Larmor frame

- If the particle begins its trajectory offset in $x$ $(x = x_0)$, but not in $y$, and with no transverse momentum before the magnetic field region:




Looks like $q < 0$ for this case.

$$\theta_c = 2\theta_L$$

$$\left|\frac{d\theta_c}{dt}\right| = |\omega_c| = \left|2\frac{d\theta_L}{dt}\right| \equiv 2\,|\omega_L|$$

$$\begin{aligned} x_L &= 2 \times \frac{x_0}{2}\cos\theta_L \\ &= x_0\cos(\omega_L t) \\ y_L &= 0 \end{aligned}$$

→ They are solutions of the following simple harmonic oscillators:

$$\ddot{x}_L + \omega_L^2 x_L = 0, \quad \ddot{y}_L + \omega_L^2 y_L = 0$$

# 2.4 Motion in a uniform electric field

- For uniform electric field $\boldsymbol{E} = E_0\hat{z}$, with $\boldsymbol{B} = 0$:

$$\frac{dp_z}{dt} = qE_0, \quad \frac{d\mathbf{p}_\perp}{dt} = 0$$

$$E_0\hat{z} = -\frac{\partial \phi_e}{\partial z}\hat{z}$$

- In terms of the potential energy, the Hamiltonian (total canonical energy) is given by

$$H = \gamma m_0 c^2 + q\phi_e = \gamma m_0 c^2 - qE_0 z$$

- Because the Hamiltonian is independent of time, it is a constant of motion:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$$

$$const. = H|_{z=0} = \gamma(0)m_0c^2 = \gamma(z)m_0c^2 - qE_0$$

$$\gamma(z) = \frac{H|_{z=0}}{m_0c^2} + \frac{qE_0}{m_0c^2}z = \gamma|_{z=0} + \frac{qE_0}{m_0c^2}z$$

- Linear increase in mechanical energy $U = \gamma m_0 c^2 = T + m_0 c^2$ (see page 18 of FOBP):

$$U^2 = p^2c^2 + (m_0c^2)^2 \longrightarrow dU = vdp \longrightarrow \frac{dp}{dt} = qE_0 = \frac{dU}{dz}$$

Accelerating gradient [MeV/m]
= Change in momentum per unit time
= Spatial energy gradient

# 2.4 Motion in a uniform electric field (cont'd)

- Other relevant dynamical variables can be derived from knowledge of $\gamma(z)$:

$$p(z) = \beta\gamma m_0 c = \sqrt{(\gamma^2(z) - 1)} \times m_0 c$$

$$v(z) = \beta c = \frac{p(z)c^2}{U(z)} = c\sqrt{1 - \frac{1}{\gamma^2(z)}}$$

- We can also explore acceleration from the point of view of explicit time dependence:

$$\frac{d\gamma}{dt} = \beta(1-\beta^2)^{-3/2}\frac{d\beta}{dt} = \beta\gamma^3\frac{d\beta}{dt}$$

$$\frac{dp_z}{dt} = m_0 c\frac{d(\beta_z\gamma)}{dt} = m_0 c\left[\gamma + \beta_z^2\gamma^3\right]\frac{d\beta_z}{dt} \simeq \gamma^3 m_0 c\frac{d\beta_z}{dt} = qE_0$$

$\beta_z \simeq \beta$

$$\frac{dv_z}{dt} = \frac{qE_0}{\gamma^3 m_0}$$

Longitudinal mass

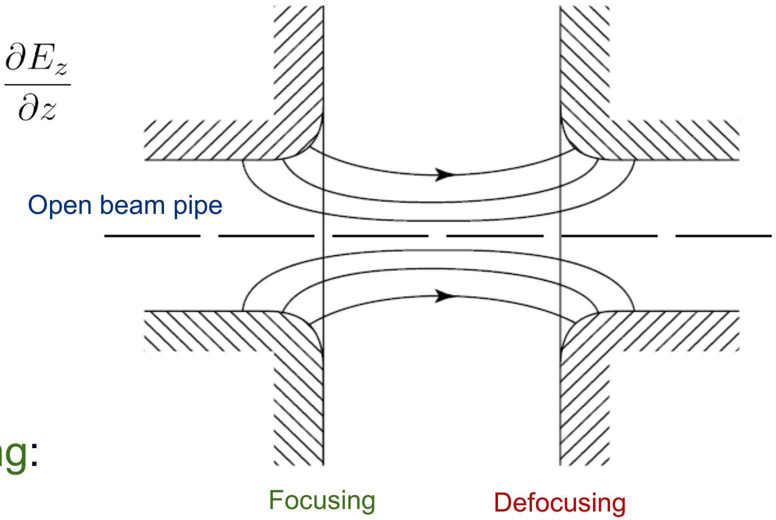
# [Note] Edge effects

- For the case of entry into a uniform electric field with azimuthal symmetry:

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\epsilon_0} = 0 = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\rho) + \cancel{\frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi}} + \frac{\partial E_z}{\partial z}$$

- Near the axis:

$$E_\rho \simeq -\frac{\rho}{2}\left(\frac{\partial E_z}{\partial z}\right)_{\rho=0}$$




- Inward force (focusing momentum kick) when entering:

$$\Delta p_\perp \simeq q\int_{t_1}^{t_2} E_\rho dt = \frac{q}{v}\int_{z_1}^{z_2} E_\rho dz = -\frac{q\rho}{2v}\int_{z_1}^{z_2}\left(\frac{\partial E_z}{\partial z}\right)_{\rho=0} dz = -\frac{q\rho}{2v}\left[E_z(z_2) - \cancel{E_z(z_1)}\right] = -\frac{q\rho}{2v}E_0$$

Kick is $-\hat{\rho}$ direction for $qE_0 > 0$

- Outward force (defocusing momentum kick) when exiting:

- No exact cancellation between focusing and defocusing momentum kicks:
  - Fields vary in time as the particles cross the gap. For longitudinal stability, the field is rising when the reference(synchronous) particle is injected. A field in the second half that is higher than the field in the first half, resulting in a net defocusing force: RF-defocusing force (important for ion linacs).
  - The particle velocity increases and radial position changes, while the particle crosses the gap: more important in electron linacs.

# 2.5 Motion in quadrupole fields

- Field free ($\mathbf{J} = 0$) vacuum region ($\mu = \mu_0$):

Sometimes (-) sign is omitted for simplicity

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = 0 \quad \longrightarrow \quad \mathbf{B} = -\nabla\psi, \quad \nabla^2\psi = 0$$

- In the limit of a device long compared to its transverse dimensions:

$$\nabla^2\psi \approx \nabla_\perp^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

- The solution of the above equation are of a form that is well behaved on axis (by separation of variables):

$$\psi = \sum_{n=1}^{\infty} a_n\rho^n\cos(n\phi) + b_n\rho^n\sin(n\phi)$$

→ Be careful ! Index convention (n) differs in US and Europe, and by authors and textbooks

# 2.5 Motion in quadrupole fields (cont'd)

- For $n = 1$:

$$\psi_1 = a_1 \rho \cos(\phi) + b_1 \rho \sin(\phi) = a_1 x + b_1 y$$

→ Equipotential surfaces form lines

$$\mathbf{B}_1 = -\nabla \psi_1 = -\left( \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} \right) \psi_1 = -a_1 \hat{x} - b_1 \hat{y}$$

Skew dipole Dipole

- For $n = 2$:

$$\psi_2 = a_2 \rho^2 \cos(2\phi) + b_2 \rho^2 \sin(2\phi) = a_2 (x^2 - y^2) + 2 b_2 x y$$

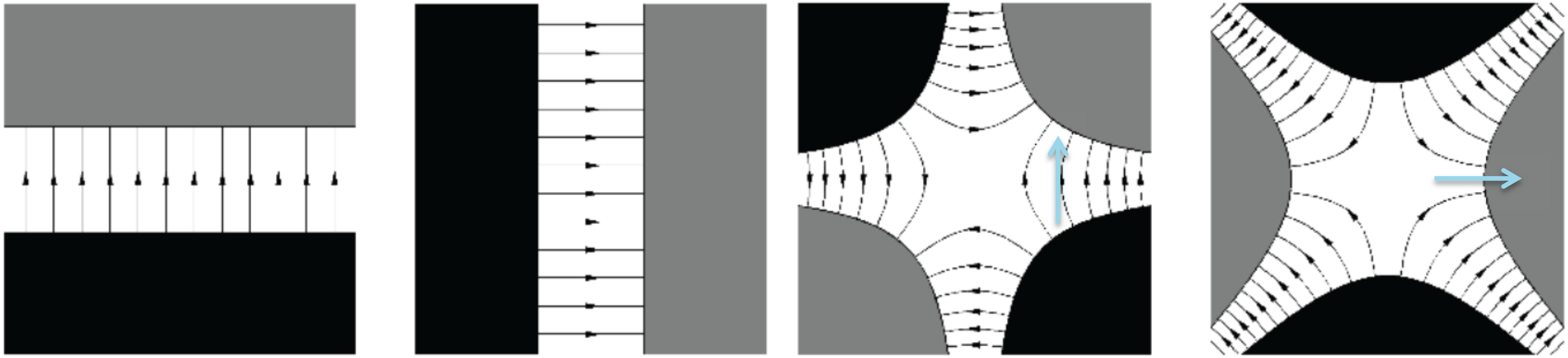
→ Equipotential surfaces form hyperbolae

$$\mathbf{B}_2 = -\nabla \psi_2 = -\left( \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} \right) \psi_2 = 2a_2(-x\hat{x} + y\hat{y}) - 2b_2(y\hat{x} + x\hat{y})$$

Skew quadrupole Quadrupole

Dipole and Skew dipole

Quadrupole and Skew quadrupole



# 2.5 Motion in quadrupole fields (cont'd)



- Force due to quadrupole fields:

Please check whether the sign is correct in Eq. (2.44) of FOBP

$$\mathbf{F}_\perp = qv_z \hat{z} \times \mathbf{B}_2 = -2qv_z b_2 (y\hat{y} - x\hat{x})$$

- Meaning of the coefficient $b_2$: Measure of field gradient

$$-2b_2 = \left.\frac{\partial B_x}{\partial y}\right|_{(0,0)} = \left.\frac{\partial B_y}{\partial x}\right|_{(0,0)} \equiv B'$$

- Transverse equations of motion for a momentum $p_0$, assuming paraxial motion near the $z$-axis:

$$\begin{aligned} x'' &= \frac{F_x}{\gamma m_0 v_0^2} = \frac{+2qv_z b_2 x}{\gamma m_0 v_0^2} = -\frac{qB'}{p_0} x \\ y'' &= \frac{F_y}{\gamma m_0 v_0^2} = \frac{-2qv_z b_2 y}{\gamma m_0 v_0^2} = +\frac{qB'}{p_0} y \end{aligned}$$

- In standard oscillator form:

$$x'' + \kappa_0^2 x = 0, \quad y'' - \kappa_0^2 y = 0$$

- Here, the square wave number is sometimes known as the focusing strength:

$$\kappa_0^2 \equiv \frac{qB'}{p_0} = K$$

# 2.5 Motion in quadrupole fields (cont'd)

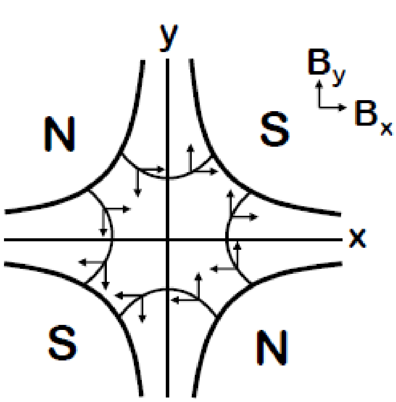
- For $\kappa_0^2 > 0$, one has simple harmonic oscillation in $x$ (around $x=0$), and the motion in $y$ is hyperbolic.

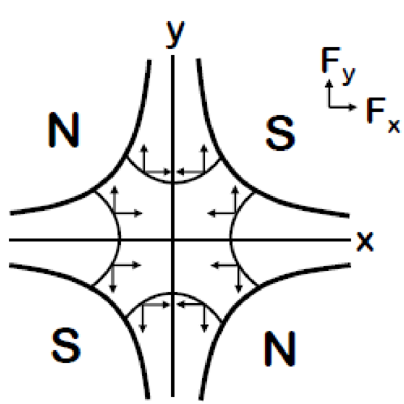
$$x = x_0 \cos\left[\kappa_0(z-z_0)\right] + \frac{x_0'}{\kappa_0} \sin\left[\kappa_0(z-z_0)\right] \qquad \text{with} \quad x(z_0) = x_0, \quad x'(z_0) = x_0'$$

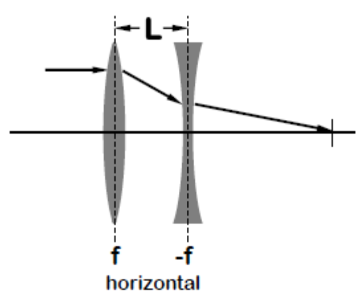
$$y = y_0 \cosh\left[\kappa_0(z-z_0)\right] + \frac{y_0'}{\kappa_0} \sinh\left[\kappa_0(z-z_0)\right] \qquad \text{with} \quad y(z_0) = y_0, \quad y'(z_0) = y_0'$$

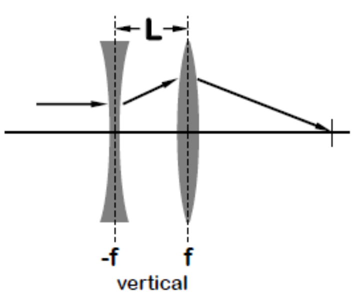
- For $\kappa_0^2 < 0$, the motion is simple harmonic(oscillatory) in $y$, and hyperbolic(unbounded) in $x$.

- Focusing with quadrupoles alone can only be accomplished in one transverse direction at a time. Ways of circumventing this apparent limitation in achieving transverse stability, by use of alternating gradient focusing.



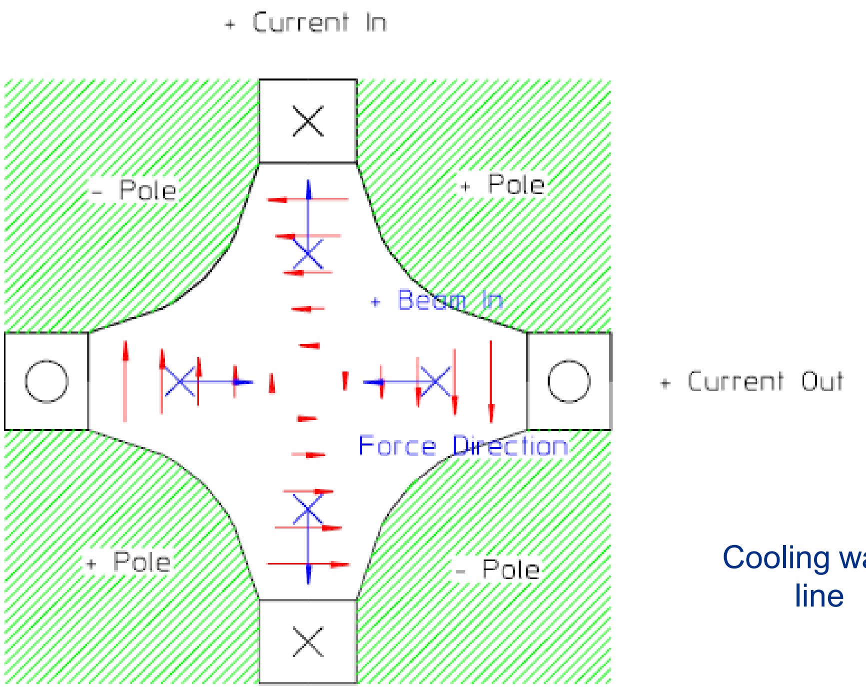



# [Example]

- Field varies linearly




$$B_x = B'y, \quad B_y = B'x \qquad (B' < 0 \;\; \text{for this Fig.})$$

$$\mathbf{B} = B'\left(y\hat{x} + x\hat{y}\right)$$

$$\mathbf{A} = -\frac{1}{2}B'\left(x^2 - y^2\right)\hat{z}$$

The in and out conductors should be placed close to each other so that longitudinal fields are minimized.

Cooling water line

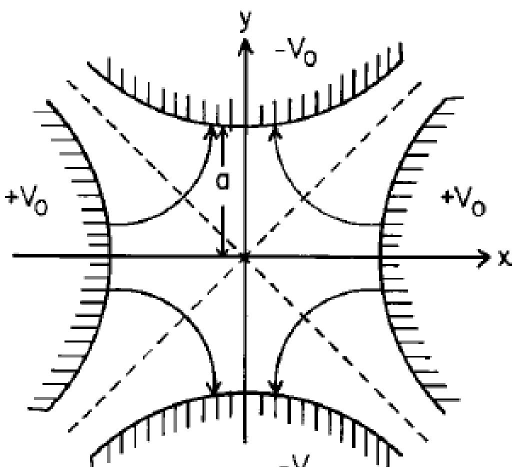
A standard technique for insulating magnet coils is to use epoxy resin, reinforced with fiberglass.

# [Note] Electric quadrupole

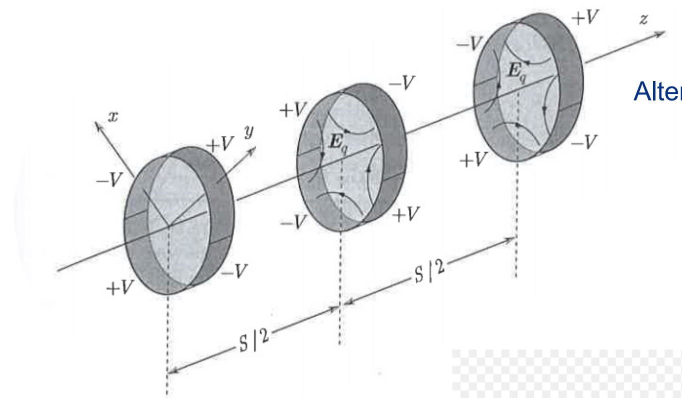
- The commonly encountered level of 1 T static magnetic field is equivalent to a 299.8 MV/m static electric field in force for a relativistic *(v ≈ c)* charged particle.

- This electric field exceeds typical breakdown limits on metallic surfaces by nearly two orders of magnitude, giving partial explanation to the predominance of magnetostatic devices over electrostatic devices for manipulation of charged particle beams.

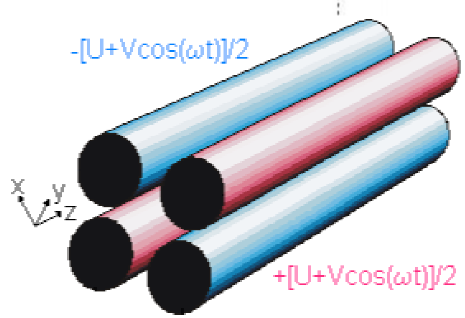
- Therefore, the transverse electric field quadrupole is found mainly in very low energy applications.



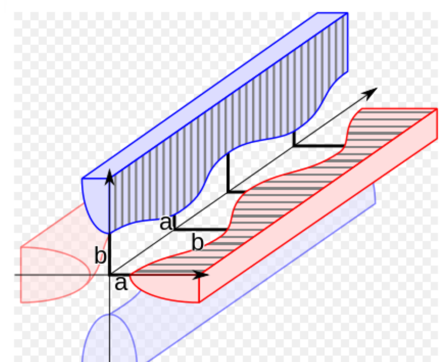

Hyperbolic surfaces rotated by 45 degrees from magnetic case




Alternating in space




Alternating in time




Alternating in time + longitudinal modulation

# 2.6 Motion in parallel, uniform E & B fields

- In case of a uniform electric field and a parallel uniform magnetic field:

$$\frac{dp_z}{dt} = qE_0, \quad \frac{d\mathbf{p}_\perp}{dt} = q(\mathbf{v}\times\mathbf{B}) = \frac{qB_0}{\gamma m_0}(\mathbf{p}_\perp \times \hat{z})$$

  - These equations are coupled by the presence of $\gamma = \left[1-(v_\perp^2 + v_z^2)/c^2\right]^{-1/2}$
  - Amplitude of the perpendicular momentum is invariant:

$$\mathbf{p}_\perp \cdot \frac{d\mathbf{p}_\perp}{dt} = \frac{d}{dt}\left(\frac{1}{2}\mathbf{p}_\perp^2\right) = 0 \longrightarrow p_\perp = const.$$

- The energy can be found similar to Eq. (2. 28) of FOBP:

$$\begin{aligned} U(z) = \gamma(z)m_0c^2 = H|_{z=0} + qE_0 z &= \sqrt{(p^2|_{z=0})c^2 + (m_0c^2)^2} + qE_0 z \\ &= \sqrt{(p_\perp^2 + p_z^2|_{z=0})c^2 + (m_0c^2)^2} + qE_0 z \\ &\to \sqrt{(p_z^2|_{z=0})c^2 + (m_0c^2)^2} + qE_0 z \end{aligned}$$

Paraxial approximation

- Normalized acceleration gradient:

$$\gamma' = \frac{d\gamma}{dz} = \frac{qE_0}{m_0c^2}$$

# 2.6 Motion in parallel, uniform E & B fields (cont'd)

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- Transverse equations of motion in the Larmor frame:

$$x_L' = \frac{dx_L}{dz} = \frac{v_{xL}}{v_z} = \frac{p_{xL}}{p_z} = \frac{p_\perp \cos\theta_L}{p_z}, \quad y_L' = \frac{dy_L}{dz} = \frac{v_{yL}}{v_z} = \frac{p_{yL}}{p_z} = \frac{p_\perp \sin\theta_L}{p_z}$$

  - We expect the maximum angle in $x_L$ and $y_L$ to be damped by the acceleration (as $p_z$ is increasing), on a time scale longer than the relevant Larmor oscillation period: adiabatic damping

- Equation of motion in $x_L$:

$$x_L'' = \frac{d}{dz}\left(\frac{p_{xL}}{p_z}\right) = -\frac{p_{xL}p_z'}{p_z^2} + \frac{p_{xL}'}{p_z} \longrightarrow x_L'' + \frac{p_{xL}p_z'}{p_z^2} - \frac{p_{xL}'}{p_z} = 0$$

  - Last term is

$$p_{xL}' = (p_\perp \cos\theta_L)' = -p_\perp \sin\theta_L \frac{d\theta_L}{dz} = p_\perp \sin\theta_L \left(\frac{qB_0}{2\beta\gamma m_0 c}\right)$$

$$\frac{p_\perp \sin\theta_L}{p_z} = y_L' = (r_L \sin\theta_L)' = r_L \cos\theta_L \frac{d\theta_L}{dz} = -x_L \left(\frac{qB_0}{2\beta\gamma m_0 c}\right)$$

- Using the paraxial approximation ($p_z \cong p$) and assuming highly relativistic motion ($\beta \cong 1$):

$$\longrightarrow x_L'' + \frac{(\beta\gamma)'}{\beta\gamma}x_L' + \left(\frac{qB_0}{2\beta\gamma m_0 c}\right)^2 x_L = 0 \longrightarrow x_L'' + \frac{\gamma'}{\gamma}x_L' + \left(\frac{b\gamma'}{2\gamma}\right)^2 x_L = 0$$

where

$$\gamma' \equiv qE_0/m_0c^2, \quad b \equiv B_0 c/E_0$$

# 2.6 Motion in parallel, uniform E & B fields (cont'd)

- The solution to the homogeneous equation:

$$x_L(z) = x_{L,0}\cos\left[\frac{b}{2}\ln\left(\frac{\gamma(z)}{\gamma_0}\right)\right] + \frac{2\gamma_0}{b\gamma'}x'_{L,0}\sin\left[\frac{b}{2}\ln\left(\frac{\gamma(z)}{\gamma_0}\right)\right]$$

$$x'_{\mathrm{L}}(z) = -x_{\mathrm{L},0}\frac{b\gamma'}{2\gamma}\sin\left[\frac{b}{2}\ln\left(\frac{\gamma(z)}{\gamma_0}\right)\right] + \frac{\gamma_0}{\gamma}x'_{\mathrm{L},0}\cos\left[\frac{b}{2}\ln\left(\frac{\gamma(z)}{\gamma_0}\right)\right]. \quad (2.60)$$

where the initial offset, angle, and Lorentz factor are

$$x_{L,0} = x_L(z=0), \quad x'_{L,0} = x'_L(z=0), \quad \gamma_0 = \gamma(z=0)$$

- In the paraxial approximation, we often replace a momentum ($p_x$) with the angle ($x'$) in the phase-space plot. We call this a trace-space plot (see page 24 of FOBP).
- In this case, we make a plot of the trajectory in the $(x, x')$ plane. This does not introduce complication in understanding the motion at constant values of $p_z$, because we can always recover the transverse momentum by using $p_x = p_z x' \cong \beta\gamma m_0 x'$.
- If longitudinal acceleration occurs, however, the angle is diminished and an apparent damping of the motion is observed: so-called adiabatic damping

# [Note] Action in trace space

- Simple harmonic motion is associated with two invariants: the angular frequency and the value of the Hamiltonian (the total oscillator energy).

$$H = \frac{1}{2m}\left[p_x^2 + m^2\omega^2 x^2\right] = J_x \omega$$

$$J_x = \frac{\oint p_x dx}{2\pi} = \frac{A}{2\pi} = \frac{1}{2} x_{\max} p_{x,\max}$$

- For the motion in parallel, uniform electric and magnetic fields, one can see that (setting $x'_{L,0} = 0$ for simplicity)

$$J_{x,\text{trace}} = \frac{x_{L,\max} x'_{L,\max}}{2} = x_{L,0}^2 \frac{\gamma'}{\gamma} \frac{b}{2} \propto \frac{1}{\gamma} \to \frac{1}{\beta\gamma}$$

# 2.7 Motion in crossed uniform E&B fields

- Perpendicular equation of motion:

$$\frac{d\mathbf{p}_\perp}{dt} = \frac{d\gamma m_0 \mathbf{v}_\perp}{dt} = q(\mathbf{E}_\perp + \mathbf{v}_\perp \times \mathbf{B})$$

- The effect of crossed uniform electric and magnetic fields can be accounted for by writing

$$\mathbf{v}_\perp = \mathbf{v}' + \mathbf{v}_d (= const.)$$

$$\frac{d\gamma m_0 \mathbf{v}'}{dt} = q(\mathbf{E}_\perp + \mathbf{v}_d \times \mathbf{B} + \mathbf{v}' \times \mathbf{B})$$

- We'd like to make

$$\frac{d\gamma m_0 \mathbf{v}'}{dt} = q\mathbf{v}' \times \mathbf{B}, \quad \mathbf{E}_\perp + \mathbf{v}_d \times \mathbf{B} = 0$$

Cyclotron motion in the moving frame

E cross B drift

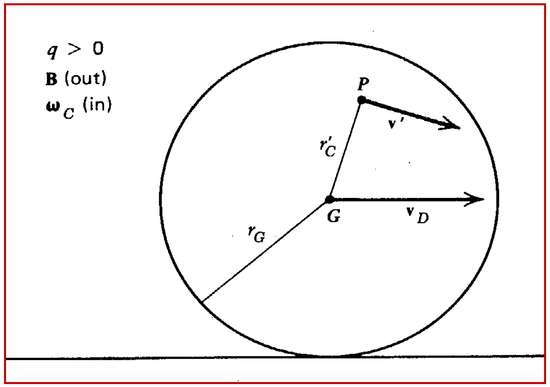
- By taking cross-product with ***B***,

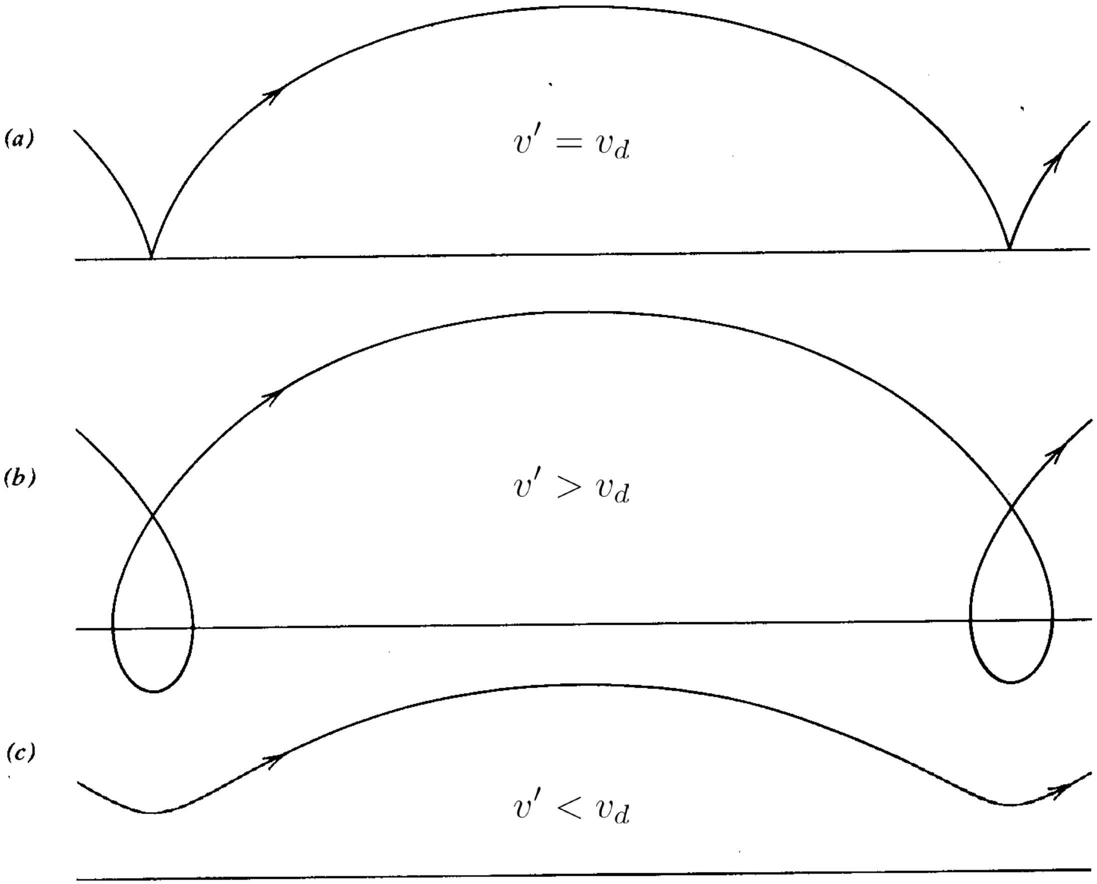
$$\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \quad v_d = \frac{E_\perp}{B} < c$$

# [Example]

- In the non-relativistic limit:

$$\frac{r'_c}{r_G} = \frac{(m_0 v'/qB)}{(v_d \tau_c/2\pi)} = \frac{(m_0 v'/qB)}{v_d \omega_c} = \frac{v'}{v_d} = \frac{B}{E_\perp} v'$$



# [Note] Lorentz transformation

- Notation with respect to the ExB drift motion (not with respect to the B-field as in the previous notation):

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}_{\perp}\right), \quad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2}\times\mathbf{E}_{\perp}\right)$$

- Case 1: $\mathbf{v}_d = \dfrac{\mathbf{E}\times\mathbf{B}}{B^2}, \quad v_d = \dfrac{E_{\perp}}{B} < c$

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} = 0, \quad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} = 0$$

$$\begin{aligned}
\mathbf{E}'_{\perp} &= \gamma_d\left(\mathbf{E}_{\perp} + \mathbf{v}_d\times\mathbf{B}_{\perp}\right) \\
&= \gamma_d\left[\mathbf{E} + \left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)\times\mathbf{B}\right]_{\perp} \\
&= \gamma_d\left[\mathbf{E} - \frac{1}{B^2}\left\{\mathbf{E}B^2 - \mathbf{B}(\mathbf{E}\cdot\mathbf{B})\right\}\right]_{\perp} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}'_{\perp} &= \gamma_d\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}_d}{c^2}\times\mathbf{E}_{\perp}\right) \\
&= \gamma_d\left[\mathbf{B} - \frac{1}{c^2}\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)\times\mathbf{E}\right]_{\perp} \\
&= \gamma_d\left[\mathbf{B} - \frac{1}{c^2B^2}\left\{\mathbf{B}E^2 - \mathbf{E}(\mathbf{E}\cdot\mathbf{B})\right\}\right]_{\perp} \\
&= \gamma_d\left[1 - \frac{1}{c^2}\frac{E^2}{B^2}\right]\mathbf{B}_{\perp} \\
&= \gamma_d\left[1 - \frac{v_d^2}{c^2}\right]\mathbf{B}_{\perp} = \frac{\mathbf{B}_{\perp}}{\gamma_d}.
\end{aligned}$$

  – In the moving frame, the only field acting is a static magnetic field. The particle has cyclotron motion.

# [Note] Lorentz transformation (cont'd)

- Case 2: $\mathbf{v}_d = \dfrac{\mathbf{E}\times\mathbf{B}}{E^2}, \quad v_d = \dfrac{E_\perp}{B} > c$

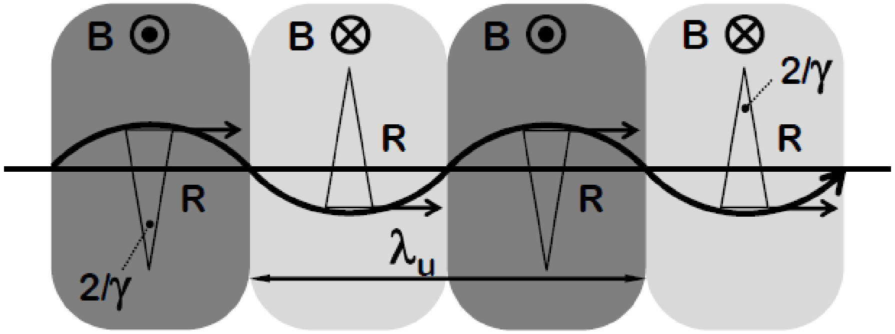
$$\mathbf{E}'_{||} = \mathbf{E}_{||} = 0, \quad \mathbf{B}'_{||} = \mathbf{B}_{||} = 0$$

$$\mathbf{E}'_\perp = \frac{1}{\gamma_d}\mathbf{E}_\perp \qquad \mathbf{B}'_\perp = 0$$

  - In the moving frame, the only field acting is a static electric field. The particle has hyperbolic motion with ever-increasing velocity.

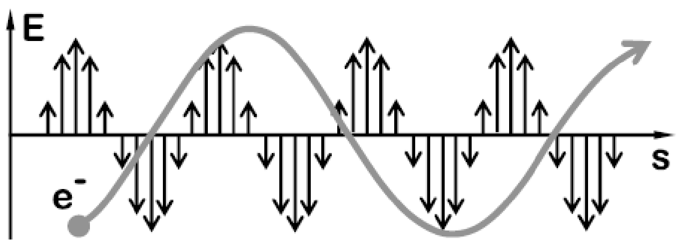
# 2.8 Motion in a periodic magnetic field

- Basics of FEL:



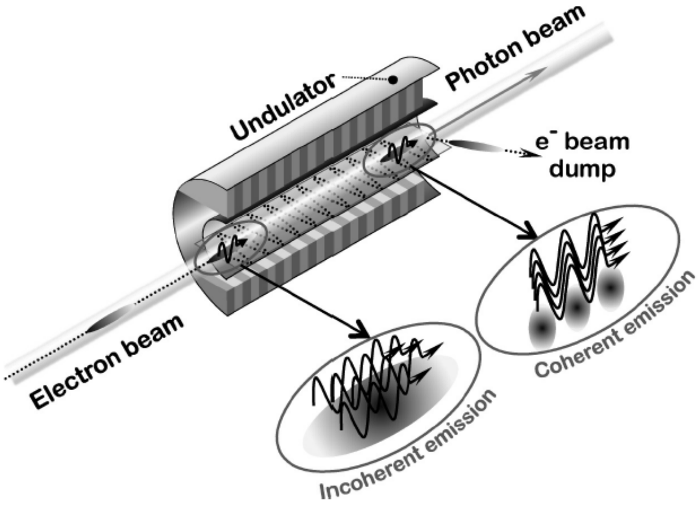


EM Radiation in a sequence of bending magnet




$$\mathbf{E} \cdot \mathbf{v} \neq 0$$

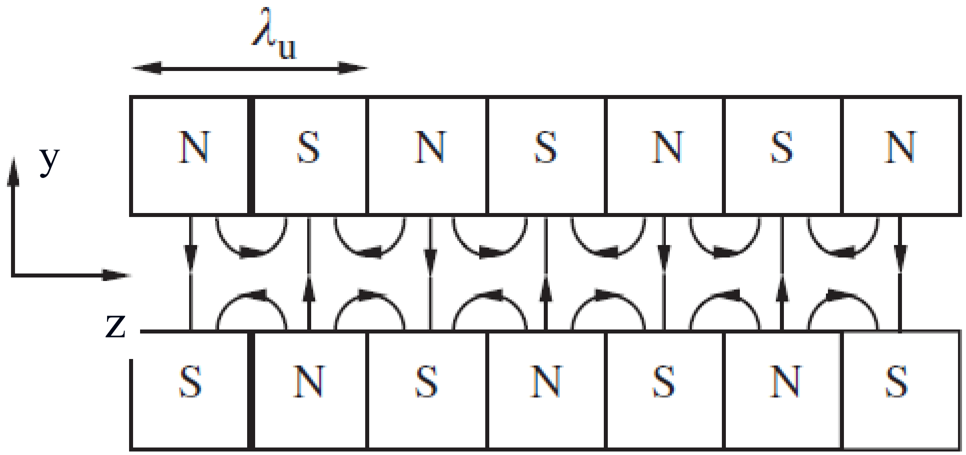
Net energy transfer between EM wave and electrons




Micro-bunching with EM wavelength

# 2.8 Motion in a periodic magnetic field (cont'd)

- The periodic, vertically polarized magnetic field (planar undulator configuration)




$$\mathbf{B} = B_0 \sin(k_u z)\hat{y}$$

Approximately correct if $k_u y \ll 1$

$$\mathbf{A} = -\frac{B_0}{k_u}\cos(k_u z)\hat{x} \longrightarrow \mathbf{A} = -\frac{B_0}{k_u}\cos(k_u z)\cosh(k_u y)\hat{x}$$

To satisfy Maxwell equation (Prob. 2. 15)

# 2.8 Motion in a periodic magnetic field (cont'd)

- Canonical momenta:

$$\begin{aligned} p_{\mathrm{c},x} &= \beta_x \gamma m_0 c + qA_x = \beta_x \gamma m_0 c - q\frac{B_0}{k_\mathrm{u}} \cos(k_\mathrm{u} z) \cosh(k_\mathrm{u} y), \\ p_{\mathrm{c},y} &= \beta_y \gamma m_0 c, \\ p_{\mathrm{c},z} &= \beta_z \gamma m_0 c, \end{aligned} \qquad (2.66)$$

- Relativistically correct Hamiltonian with $\phi_e = 0$:

$$H = \sqrt{\left(p_{\mathrm{c},x} + q\frac{B_0}{k_\mathrm{u}} \cos(k_\mathrm{u} z) \cosh(k_\mathrm{u} y)\right)^2 c^2 + p_{\mathrm{c},y}^2 c^2 + p_{\mathrm{c},y}^2 c^2 + (m_0 c^2)^2}, \qquad (2.67)$$

$p_{c,z}^2 c^2$

# 2.8 Motion in a periodic magnetic field (cont'd)

- Viewing $z$ as the independent variable (kind of a canonical transformation):

$$G = -p_{z,c} = \sqrt{H^2 - (p_{c,y} - qA_y)^2c^2 - (p_{c,x} - qA_x)^2c^2 - (m_0c^2)^2} \; /c$$
$$= \sqrt{U^2 - (p_{c,y} - qA_y)^2c^2 - \left(p_{c,x} + q\frac{B_0}{k_u}\cos(k_u z)\cosh(k_u y)\right)^2 c^2 - (m_0c^2)^2}, \; /c \qquad (2.68)$$

  – Here we have substituted the numerical energy $U$ for the old Hamiltonian functional energy $H$.

- New Hamilton equations of motion (assuming $y = 0$ for simplicity):

$$p'_{c,x} = -\frac{\partial G}{\partial x} = 0 \longrightarrow p_{c,x} = const. = p_{x0} = \beta_x\gamma m_0 c + qA_x = \beta_x\gamma m_0 c - q\frac{B_0}{k_u}\cos(k_u z)$$
$$p'_{c,y} = -\frac{\partial G}{\partial y} = 0 \longrightarrow p_{c,y} = const. = 0 = \beta_y\gamma m_0 c$$

$$p_{c,z} = \beta_z\gamma m_0 c = \sqrt{p_0^2 - (\beta_x\gamma m_0 c)^2 - (\beta_y\gamma m_0 c)^2}$$
$$= \sqrt{p_0^2 - \left(q\frac{B_0}{k_u}\cos(k_u z) + p_{x0}\right)^2}$$

# 2.8 Motion in a periodic magnetic field (cont'd)

- New Hamilton equations of motion (assuming $y = 0$ for simplicity):

$$
\begin{aligned}
x' = \frac{\partial G}{\partial p_{c,x}} = -\frac{\partial p_{c,z}}{\partial p_{c,x}} &= -\frac{\partial}{\partial p_{c,x}}\left[\sqrt{p_0^2 - \left(q\frac{B_0}{k_u}\cos(k_u z) + p_{c,x}\right)^2}\right] \\
&\simeq -p_0\frac{\partial}{\partial p_{c,x}}\left[1 - \frac{1}{2}\left(q\frac{B_0}{p_0 k_u}\cos(k_u z) + \frac{p_{c,x}}{p_0}\right)^2\right] \\
&= \frac{qB_0}{p_0 k_u}\cos(k_u z) + \frac{p_{c,x}}{p_0} \\
&= \frac{qB_0}{p_0 k_u}\cos(k_u z) + \frac{p_{x0}}{p_0} = \frac{qB_0}{p_0 k_u}\cos(k_u z) + x_0'
\end{aligned}
$$

- With initial (evaluated before entry into the undulator field) horizontal offset and angle ($x_0$, $x_0{}'$):

$$x \simeq x_0 + x_0' z + \frac{qB_0}{p_0 k_u^2}\sin(k_u z)$$

$$\frac{qB_0}{mck_u} \equiv K \ \text{(undulator parameter)}$$

$$x \simeq x_0 + x_0' z + \frac{cK}{\gamma v k_u}\sin(k_u z)$$

  – Amplitude of the transverse motion (for $v \approx c$): $\dfrac{cK}{\gamma k_u}$

  – Maximum angle (for $v \approx c$): $\dfrac{qB_0}{p_0 k_u} = \dfrac{cK}{\gamma v} \approx \dfrac{K}{\gamma} \ll 1$

# [Note]

- If an initial error $x_0{}' \neq 0$ is not corrected, it leads eventually to a trajectory with large horizontal offset $x$.

- The longitudinal momentum (therefore the longitudinal velocity) must decrease in the undulator:

$$
\begin{aligned}
p_z &= \sqrt{p_0^2 - \left( q\frac{B_0}{k_u}\cos(k_u z) + p_{c,x} \right)^2} \\
&= p_0 \left[ 1 - \frac{1}{2}\left( \frac{qB_0}{k_u p_0}\cos(k_u z) \right)^2 \right]
\end{aligned}
\qquad p_{c,x} = p_{x0} = 0
$$

  - Averaging over a period of the motion, we have

$$
\langle p_z \rangle \simeq p_0 \left[ 1 - \left( \frac{qB_0}{2k_u p_0} \right)^2 \right]
$$

  - This slowing of the particle in its z-direction is an important effect in FELs.