Lecture 1

Basics of Electromagnetism, Special Relativity and Classical Mechanics

(Sec. 1.2 ~ 1.6 of FOBP)

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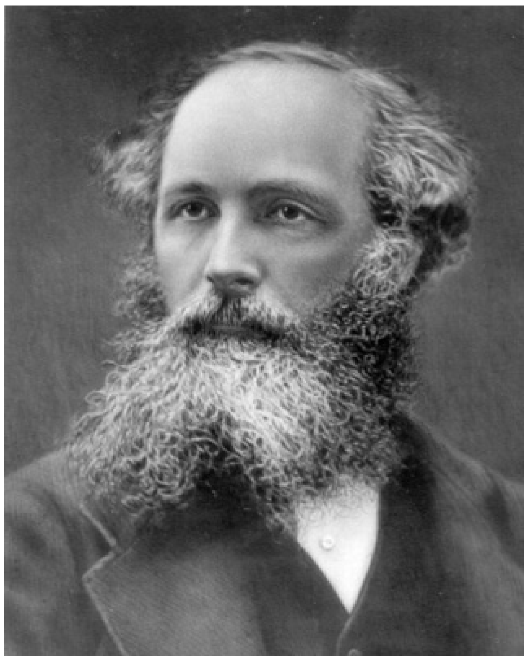
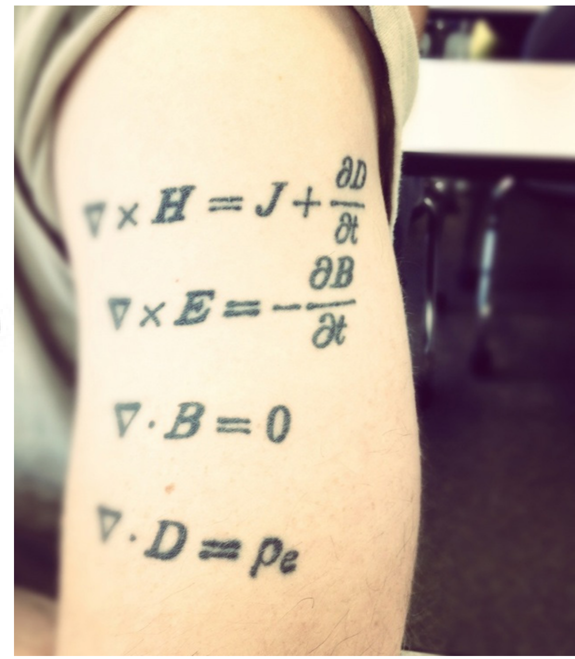
Electromagnetism





Maxwell Equations

- Classical electrodynamics is governed by the Maxwell equations. In the SI (MKS) system of units, the equations are

$$
\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}
$$

- For external sources in vacuum, the constitutive equations are

$$
\mathbf{D} = \epsilon_0 \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu_0
$$

- The equations are linear: the sum of two solutions, $\mathbf{E}_1, \mathbf{B}_1$ and $\mathbf{E}_2, \mathbf{B}_2$ is also a solution corresponding to the sum of densities $\rho_1 + \rho_2$, $\mathbf{J}_1 + \mathbf{J}_2$.

Charge and Current Densities

- The free electric charge density and current density are related by the equation of continuity, which is implicit in the Maxwell equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

- For a point charge moving along a trajectory $\boldsymbol{r} = \boldsymbol{r}_0(t)$,

$$\rho = q\delta[\mathbf{r} - \mathbf{r}_0(t)], \quad \mathbf{J} = q\mathbf{v}\delta[\mathbf{r} - \mathbf{r}_0(t)]$$

$$\mathbf{v} = \frac{d\mathbf{r}_0(t)}{dt}$$

- Note that in 3D:

$$\delta[\mathbf{r} - \mathbf{r}_0(t)] = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

  - In cylindrical coordinates:

$$\delta[\mathbf{r}-\mathbf{r}_0(t)] = \frac{1}{r}\delta(r-r_0)\delta(\theta-\theta_0)\delta(z-z_0) \xrightarrow{\text{Azimuthally symmetric case}} \frac{1}{2\pi r}\delta(r-r_0)\delta(z-z_0)$$

Scalar and Vector Potentials

- It is often convenient to express the fields in terms of the vector and the scalar potentials (two homogeneous Maxwell equations-equations with no charge or current source terms- are automatically satisfied).

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- The potentials are not uniquely specified.

$$\phi \to \phi - \frac{\partial \Lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A} + \nabla\Lambda$$

- We can choose a set of potentials to satisfy the so-called Lorentz condition.

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

- The Lorentz condition results in the symmetric and decoupled form of the inhomogeneous wave equations.

$$\nabla^2\phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho/\epsilon_0, \quad \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0\mathbf{J}$$

Coulomb Gauge

- Within a closed region of space containing no free charges, surrounded by an equipotential surface (e.g., RF cavities): Uniqueness theorem

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \rightarrow \phi = const. = 0$$

- The Lorentz condition becomes Coulomb gauge.

$$\nabla \cdot \mathbf{A} = 0$$

- The electric and magnetic fields are obtained from the vector potential alone.

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

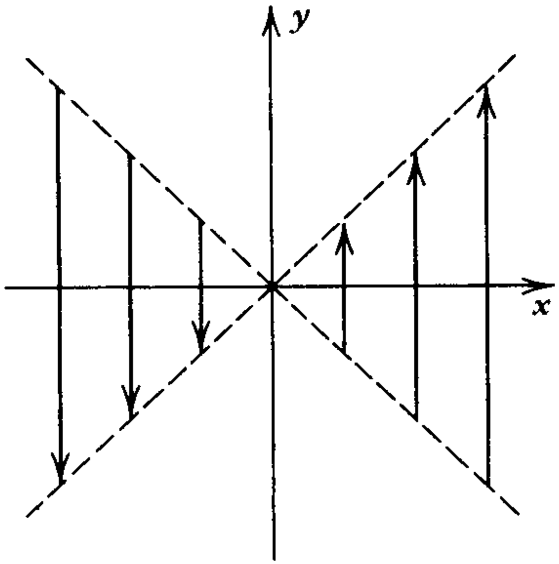
- For time-independent case (e.g., Magnets),

$$\mathbf{E} = 0, \quad \nabla^2\mathbf{A} = -\mu_0\mathbf{J} \rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'$$

Example: Uniform Magnetic Induction

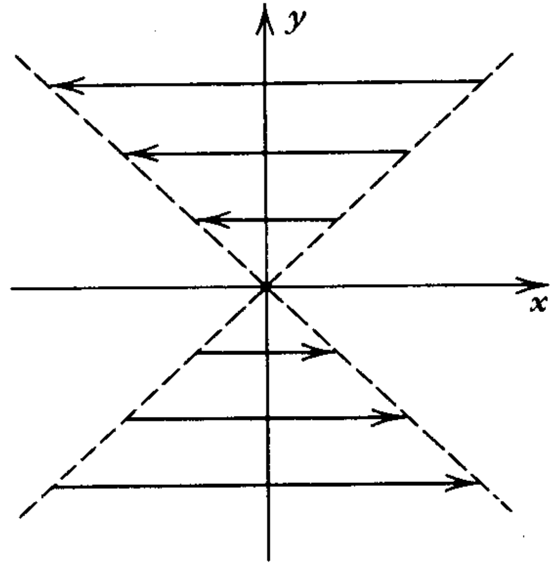
- Let us consider a uniform magnetic induction given by $\mathbf{B} = B\hat{\mathbf{z}}$ where $B = const$.

Infinite current sheets



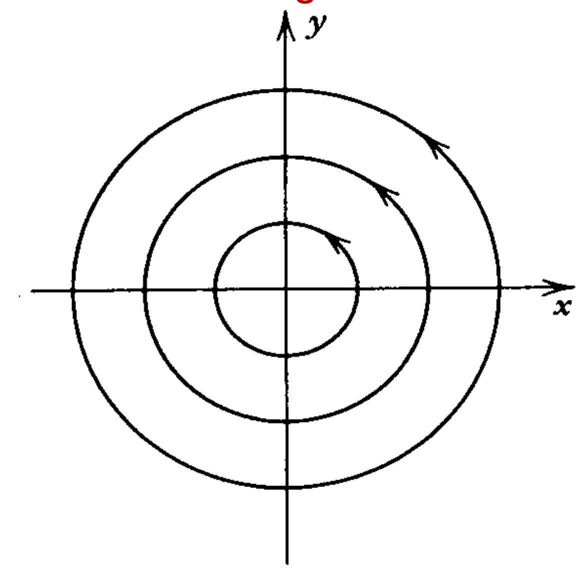
$$\mathbf{A} = (0, Bx, 0)$$

Infinite current sheets



$$\mathbf{A} = (-By, 0, 0)$$

Inside a long solenoid



$$\mathbf{A} = (-By/2, Bx/2, 0)$$

- For all three cases:

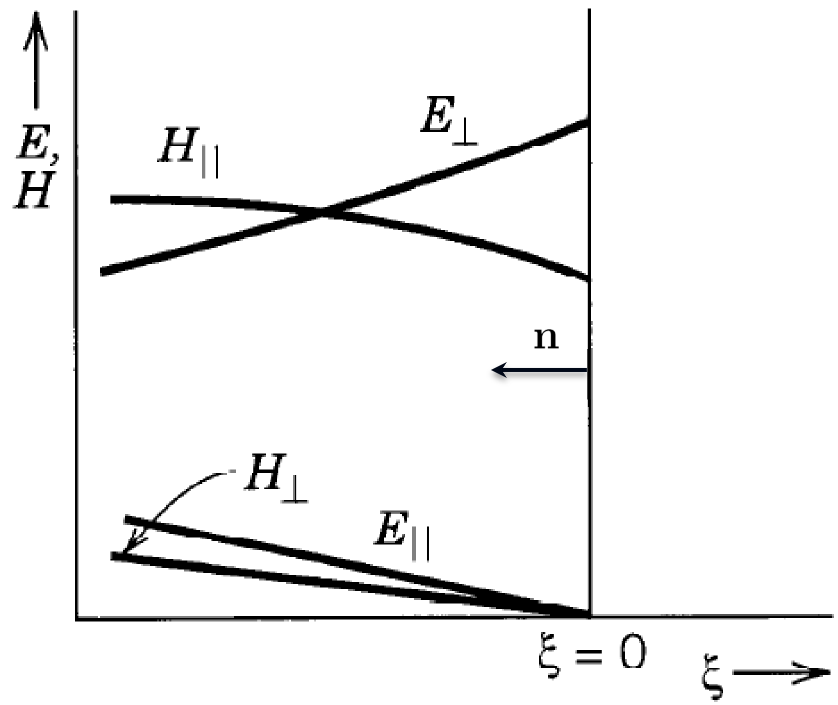
$$\nabla \cdot \mathbf{A} = 0, \quad B\hat{z} = \nabla \times \mathbf{A}, \quad \mathbf{A} \parallel \mathbf{J} \text{ by symmetry}$$

  – Uniform B fields can be produced within a given region in a variety of ways

Boundary Conditions

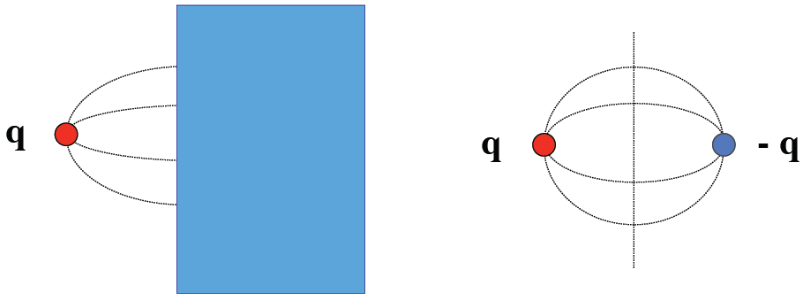
- Inside perfect conductor, all of the field vectors will be zero. If $\hat{\mathbf{n}}$ is the unit normal vector pointing outward from the surface of the conductor,

$$\hat{\mathbf{n}} \cdot \mathbf{D} = \Sigma, \quad \hat{\mathbf{n}} \times \mathbf{E} = 0, \quad \hat{\mathbf{n}} \cdot \mathbf{B} = 0, \quad \hat{\mathbf{n}} \times \mathbf{H} = \mathbf{K}$$

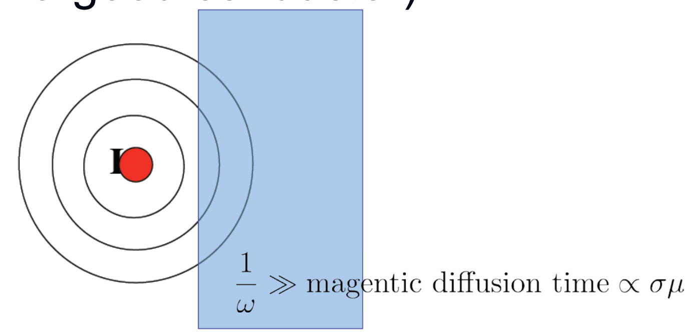


Examples of Boundary Conditions

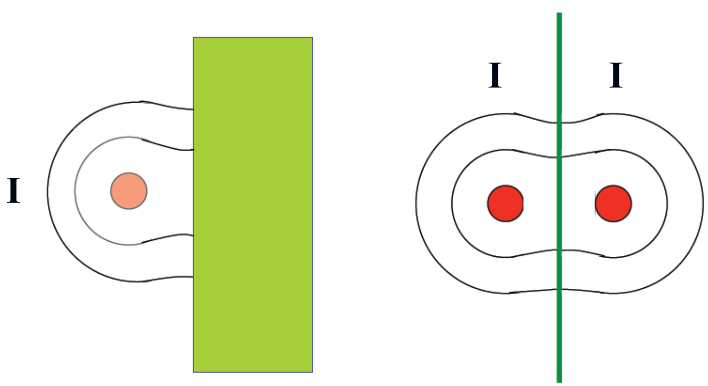
- Electric field near a good conductor:



- Static magnetic field near $\mu_r \approx 1$(even in the case of a good conductor)

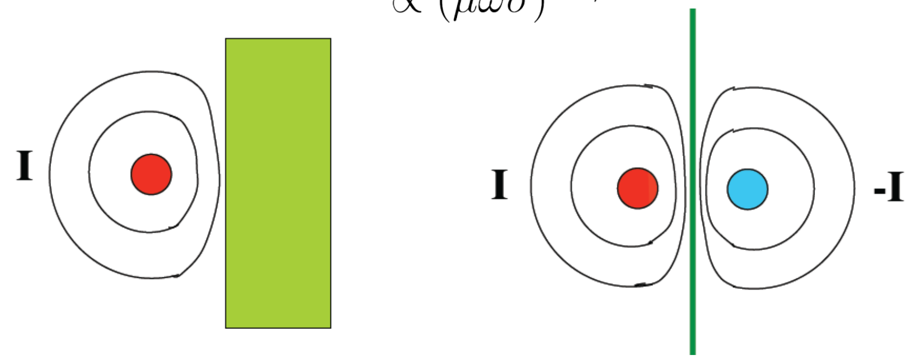


- Static magnetic field near $\mu_r \gg 1$(i.e., ferromagnetic material)



- Time-varying magnetic field near a good conductor (i.e., small skin depth):

$\propto (\mu\omega\sigma)^{-1/2}$



Energy Balance

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- The energy density of the field (energy per unit volume) is*

$$u = \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right) = \frac{\epsilon_0}{2}\left(E^2 + c^2 B^2\right)$$

- The Poynting vector gives energy flow (energy per unit area per unit time) in the electromagnetic field.

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$

- Time rate of change of electromagnetic energy within a certain volume plus the energy flowing out through the boundary surface of the volume per unit time, is equal to the negative of the total work done by the fields on the sources within the volume:

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}$$

$$\frac{\partial}{\partial t}\int_V u dv + \oint_S \mathbf{S}\cdot d\mathbf{a} = -\int_V \mathbf{J}\cdot\mathbf{E}\, dv$$

*Note that for plane EM wave in vacuum: $E = cB$ (Electric energy = Magnetic energy)

Time-Harmonic Fields

- We assume all fields and sources have a time dependence $e^{-i\omega t}$ (or $e^{j\omega t}$)

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r})e^{-i\omega t}\right]$$

Phasor

- Time-average of the products:

$$\langle \mathbf{J}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{J}^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})\right], \qquad \langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\right]$$

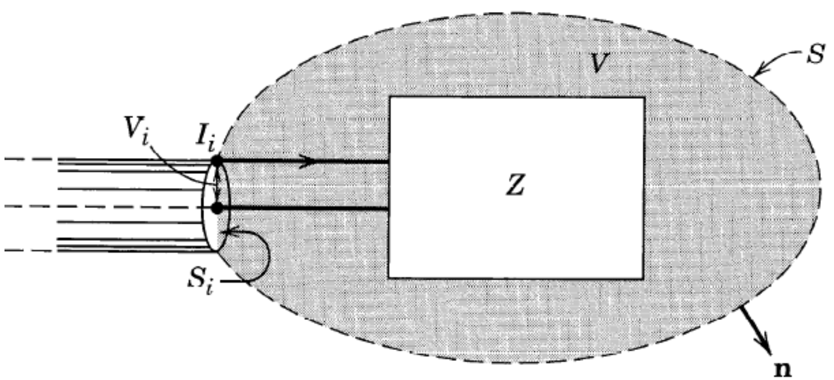
- Complex Poynting theorem:

$$\frac{1}{2}I_i^* V_i = \frac{1}{2}\underbrace{(R - iX)}_{=Z=(R+jX)}|I_i|^2 = -\frac{1}{2}\oint_{S_i}\mathbf{E}\times\mathbf{H}^*\cdot\mathbf{n}\,da$$

Complex power input

For electrical engineers

$$= \frac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}\,dv + 2i\omega\int_V (w_e - w_m)\,dv + \frac{1}{2}\oint_{S-S_i}\mathbf{E}\times\mathbf{H}^*\cdot\mathbf{n}\,da$$

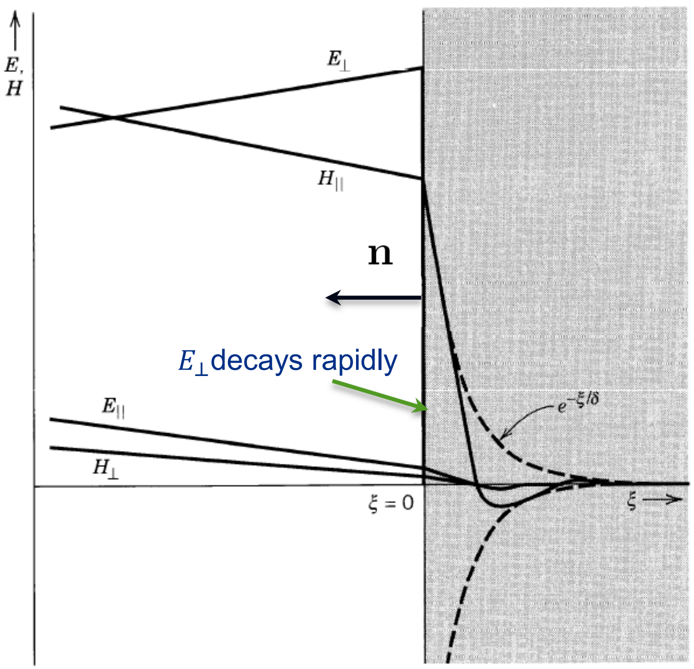


$$w_e = \frac{1}{4}(\mathbf{E}\cdot\mathbf{D}^*), \quad w_m = \frac{1}{4}(\mathbf{B}\cdot\mathbf{H}^*)$$

Skin Depth

- For a good but not perfect conductor, fields and currents are not exactly zero inside the conductor, but are confined to within a small finite layer at the surface, called the skin depth.

$$\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2}$$



- Fields inside the conductor exhibit rapid exponential decay, phase difference, magnetic field much larger than the electric field, and inside fields parallel to the surface.

$$\mathbf{H}_c = \mathbf{H}_{||} e^{-\xi/\delta} e^{i\xi/\delta}$$

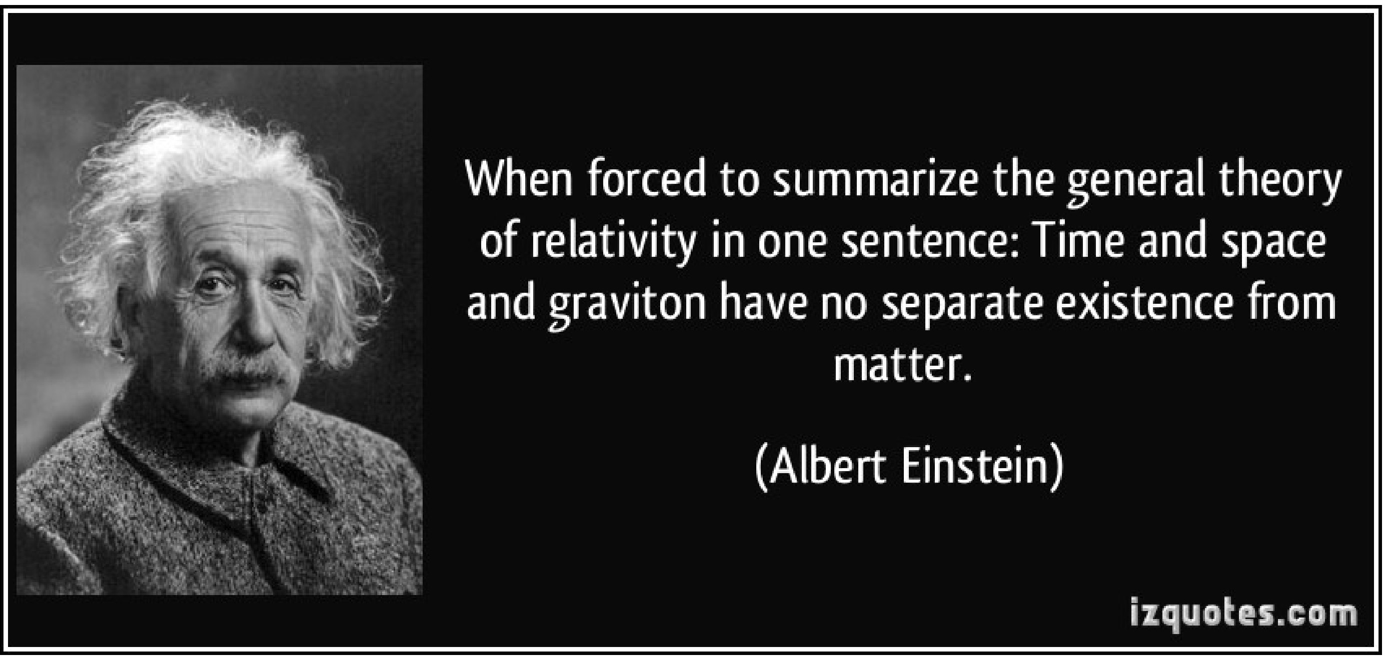
$$\mathbf{E}_c \simeq \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)(\mathbf{n}\times\mathbf{H}_{||}) e^{-\xi/\delta} e^{i\xi/\delta}$$

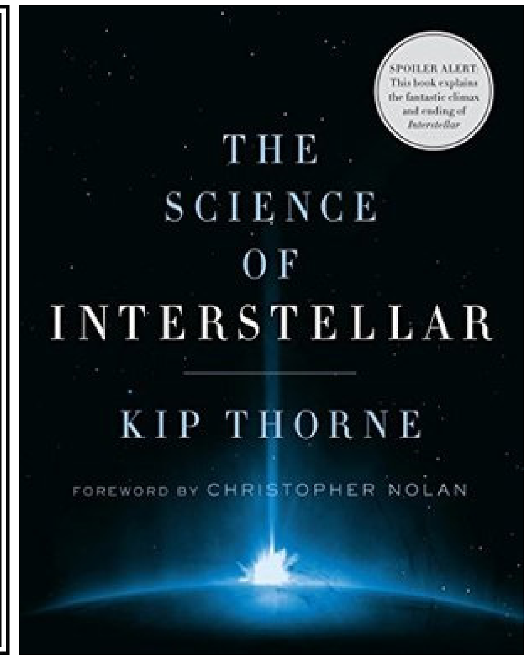
$$\mathbf{J} = \sigma \mathbf{E}_c, \ \ \mathbf{K}_{\text{eff}} = \int_0^\infty \mathbf{J}\, d\xi = \hat{\mathbf{n}} \times \mathbf{H}_{||}$$

- Time-averaged power absorbed per unit area:

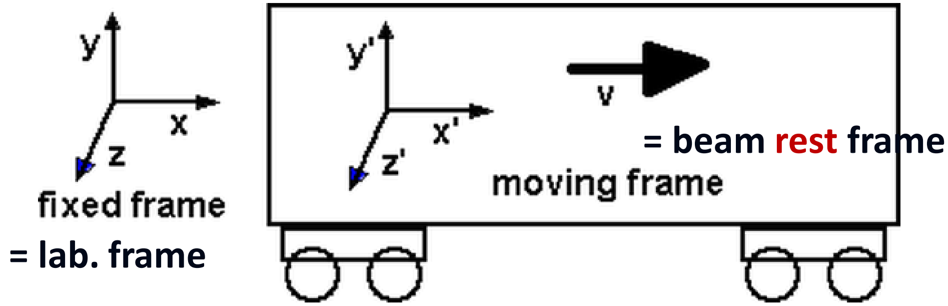
$$\frac{dP_{\text{loss}}}{da} = -\frac{1}{2}\text{Re}\left[\mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*\right] = \frac{\mu_c \omega \delta}{4}|\mathbf{H}_{||}|^2 = \frac{1}{2}\times \underbrace{\frac{1}{\sigma\delta}}_{R_s = \sqrt{\mu_c\omega/2\sigma} = \text{Surface resistance}} |\mathbf{K}_{\text{eff}}|^2$$

Relativity


When forced to summarize the general theory of relativity in one sentence: Time and space and graviton have no separate existence from matter.
(Albert Einstein)
izquotes.com


SPOILER ALERT
This book explains the fantastic climax and ending of Interstellar
THE SCIENCE OF INTERSTELLAR
KIP THORNE
FOREWORD BY CHRISTOPHER NOLAN

Inertial Frame



Galilean transformation

$$
\begin{aligned}
x' &= x - Vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

- All inertial frames are in a state of constant, rectilinear motion with respect to one another; $\boldsymbol{F} = \boldsymbol{F}'$ (the laws of the mechanics are the same)
- A non-inertial reference frame is a frame of reference that is undergoing acceleration with respect to an inertial frame.
- $\boldsymbol{F} = m\boldsymbol{a}$ holds in any coordinate system provided the term 'force' is redefined to include the so-called inertial forces.
- Lorentz transformation (only refers to transformations between inertial frames) for relativistic motions ($\gamma_b = 1/\sqrt{1 - V^2/c^2}$)

$$
\begin{aligned}
x' &= \gamma_b (x - Vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma_b \left(t - \frac{V}{c^2} x \right)
\end{aligned}
$$

Time Dilation and Lorentz Contraction

- Time interval appears to be longer to the moving observer (this is a relative concept !) than it does to the one at rest with respect to the clock.

$$\Delta t' = t_2' - t_1' = \gamma_b \left[(t_2 - t_1) - \frac{V}{c^2} \underbrace{(x_2 - x_1)}_{=0} \right] = \gamma_b \Delta t > \Delta t$$

Ex] Unstable particles such as muons should have a longer lifetime than resting ones as accelerated.(We can think the muon is at rest, the observer is moving)

- As found by the moving observer, the length (whose ends are determined simultaneously) in the direction of motion will be contracted.

$$\Delta x' = x_2' - x_1' = \gamma_b \left[(x_2 - x_1) - V(t_2 - t_1) \right]$$

$$0 = \Delta t' = t_2' - t_1' = \gamma_b \left[(t_2 - t_1) - \frac{V}{c^2}(x_2 - x_1) \right]$$

$$\Delta x' = \gamma_b \left[1 - V^2/c^2 \right] (x_2 - x_1) = \frac{\Delta x}{\gamma_b} < \Delta x$$

Ex] Longitudinal Lorentz contraction of the bunch in relativistic beams. (We can think the beam is at rest, the observer is moving)

Special Relativity Handle Acceleration?

- It is often said, erroneously, that special relativity cannot deal with accelerations because it deals only with inertial frames.

- Sometimes it is claimed that general relativity is required for these situations; if that's the case, accelerator physics must have been much more complicated!

- This is not true. We must, of course, only allow transformations between inertial frames; the frames must not accelerate.

- Special relativity treats acceleration differently from inertial frames and can deal with anything kinematic, but general relativity is required when gravitational forces are present.

Main Results of Special Relativity

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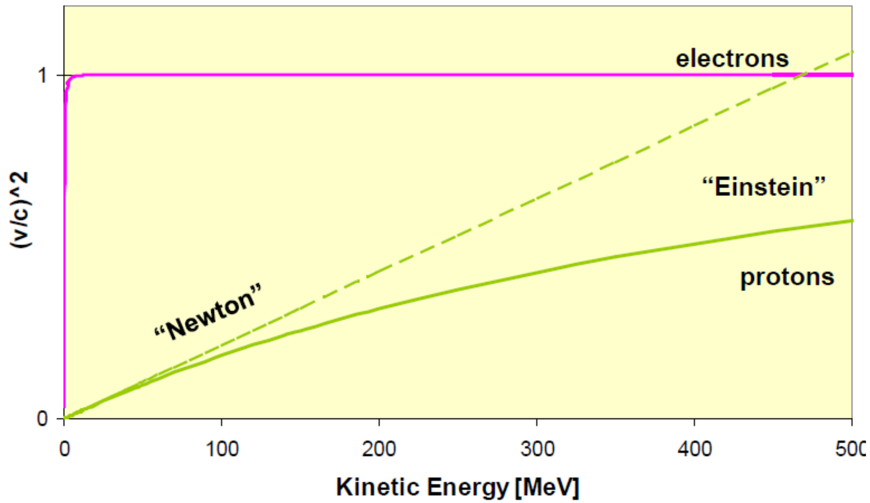
- Relativistic parameters: Don't be confused with Twiss parameters.

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1 + \frac{1}{2}\beta^2, \quad \beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$$



Energy is increased, but velocity is nearly constant

- The total energy, mechanical momentum, and kinetic energy of a rest mass m:

$$E = \gamma mc^2, \quad p = \gamma m \beta c, \quad T = (\gamma - 1)mc^2 \longrightarrow \frac{1}{2}mv^2 \ (v \ll c)$$

- The relation between total energy and momentum in the absence of EM fields:

$$E = \sqrt{p^2c^2 + (mc^2)^2}, \quad \text{or} \quad \gamma^2 = (\beta\gamma)^2 + 1$$

Energy and Mass Units

- To describe the energy of individual particles, we use the eV, the energy that a unit charge

$$e = 1.6 \times 10^{-19} \text{ Coulomb}$$

gains when it falls through a potential, $\Delta\phi$ = 1 volt.

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ Joule}$$

- We can use Einstein's relation to convert rest mass to energy units.

$$E_{rest} = mc^2$$

- For electrons,

$$E_{rest} = (9.11 \times 10^{-31} \text{ kg}) \times (3 \times 10^8 \text{ m/s})^2 = 0.511 \text{ MeV}$$

- For protons,

$$E_{rest} = (1.67 \times 10^{-27} \text{ kg}) \times (3 \times 10^8 \text{ m/s})^2 = 938 \text{ MeV}$$

Lorentz Equation and Effective Mass

- Lorentz equation: We need to consider changes of γ in time.

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m \mathbf{v}) = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\frac{d\gamma}{dt} = \beta(1-\beta^2)^{-3/2}\frac{d\beta}{dt} = \frac{\beta\gamma^3}{c}\frac{dv}{dt}$$

- Parallel and perpendicular decomposition:

$$\gamma m \left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} + \gamma m \left(\frac{d\mathbf{v}}{dt}\right)_{\perp} + m\mathbf{v}\frac{d\gamma}{dt} = \mathbf{F}_{\perp} + \mathbf{F}_{\parallel}$$

  - Parallel acceleration:

$$\mathbf{v} = v\hat{\parallel}$$

$$\gamma m \left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} + \gamma m \beta^2 \gamma^2 \left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} = \mathbf{F}_{\parallel} \longrightarrow \underbrace{\gamma^3 m}_{\text{longitudinal mass}} \left(\frac{d\mathbf{v}}{dt}\right)_{\parallel} = \mathbf{F}_{\parallel}$$

  - Perpendicular acceleration:

$$\gamma m \left(\frac{d\mathbf{v}}{dt}\right)_{\perp} = \mathbf{F}_{\perp} \longrightarrow \underbrace{\gamma m}_{\text{perpendicular mass}} \left(\frac{d\mathbf{v}}{dt}\right)_{\perp} = \mathbf{F}_{\perp}$$

Transformation of Momentum

- 4-veloity:

$$U_1 = \frac{dx_1}{d\tau} = \frac{v_x}{\sqrt{1-(v^2/c^2)}}, \quad U_2 = \frac{dx_2}{d\tau} = \frac{v_y}{\sqrt{1-(v^2/c^2)}},$$

$$U_3 = \frac{dx_3}{d\tau} = \frac{v_z}{\sqrt{1-(v^2/c^2)}}, \quad U_4 = \frac{dx_4}{d\tau} = \frac{ic}{\sqrt{1-(v^2/c^2)}}$$

where

$$v^2 = v_x^2 + v_y^2 + v_z^2, \quad d\tau = dt(1 - v^2/c^2)^{1/2} = \text{invariant} = \text{proper time}$$

$$U_1^2 + U_2^2 + U_3^2 + U_4^2 = \gamma^2 v^2 - \gamma^2 c^2 = -c^2$$

- 4-momentum:

$$(P_1, P_2, P_3, P_4) = m\,(U_1, U_2, U_3, U_4) = \left(\mathbf{p}, \frac{iE}{c}\right) \longrightarrow p^2 - \frac{E^2}{c^2} = -m^2c^2$$

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-(v^2/c^2)}} = \gamma m \mathbf{v}$$

$$P_x' = \gamma_b\left(P_x - \frac{V}{c^2}E\right), \quad P_y' = P_y, \quad P_z' = P_z, \quad E' = \gamma_b\,(E - VP_x)$$

  – Transverse momentum is invariant under this transformation.
  – Transformation rule for ordinary velocity is complicated as we need to transform both displacement and time

Transformation of Potentials and Fields

- Lorentz transformation of potentials:

$$A'_x = \gamma_b \left(A_x - \frac{V}{c^2}\phi \right), \quad A'_y = A_y, \quad A'_z = A_z, \quad \phi' = \gamma_b \left(\phi - V A_x \right)$$

- Lorentz transformation of fields: Longitudinal fields are "Lorentz invariant"

$$\mathbf{E}'_{||} = \mathbf{E}_{||}, \quad \mathbf{B}'_{||} = \mathbf{B}_{||}, \quad \mathbf{E}'_{\perp} = \gamma_b \left(\mathbf{E}_{\perp} + \mathbf{V} \times \mathbf{B}_{\perp} \right), \quad \mathbf{B}'_{\perp} = \gamma_b \left(\mathbf{B}_{\perp} - \frac{\mathbf{V}}{c^2} \times \mathbf{E}_{\perp} \right)$$

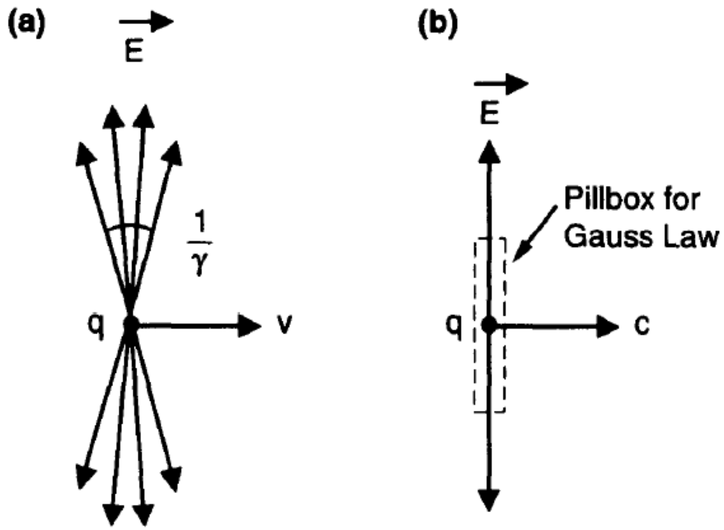
Ex] Pure electric field in beam rest frame (i.e., primed system): A pure electric field to one observer may be seen as both an electric and a magnetic field to a second observer.

$$\mathbf{E}_{\perp} = \gamma_b \left(\mathbf{E}'_{\perp} - \mathbf{V} \times \cancel{\mathbf{B}'_{\perp}} \right), \quad \mathbf{B}_{\perp} = \gamma_b \left(\cancel{\mathbf{B}'_{\perp}} + \frac{\mathbf{V}}{c^2} \times \mathbf{E}'_{\perp} \right)$$

$$\mathbf{F}_{\perp} = q \left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp} \right) \approx \frac{q\mathbf{E}_{\perp}}{\gamma_b^2}$$

Fields of Relativistic Point Charge

- The field distribution is Lorentz contracted into a thin disk perpendicular to the particle's direction of motion with an angular spread on the order of $1/\gamma$.



$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{q}{\epsilon_0}\delta[\mathbf{r} - \mathbf{r}_0(t)] = \frac{q}{\epsilon_0}\frac{\delta(r)}{2\pi r}\delta(z - ct)$$

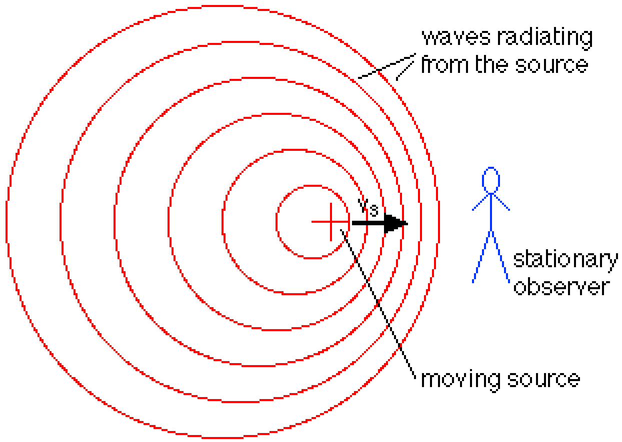
$$\longrightarrow \int 2\pi r E_r dz = \int\int \frac{q}{\epsilon_0}\frac{\delta(r)}{\cancel{2\pi r}}\delta(z - ct)\cancel{2\pi r}drdz \longrightarrow E_r = \frac{q}{2\pi\epsilon_0 r}\delta(z - ct)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0 q\mathbf{v}\delta[\mathbf{r} - \mathbf{r}_0(t)] = \mu_0 q\mathbf{v}\frac{\delta(r)}{2\pi r}\delta(z - ct)$$

$$\longrightarrow 2\pi r B_\theta = \int \mu_0 qc\frac{\delta(r)}{\cancel{2\pi r}}\delta(z - ct)\cancel{2\pi r}dr \longrightarrow B_\theta = \frac{\mu_0 q}{2\pi r}c\delta(z - ct) = \frac{E_r}{c}$$

Relativistic Doppler Shift

- The Doppler effect is modified to be consistent with the Lorentz transformation



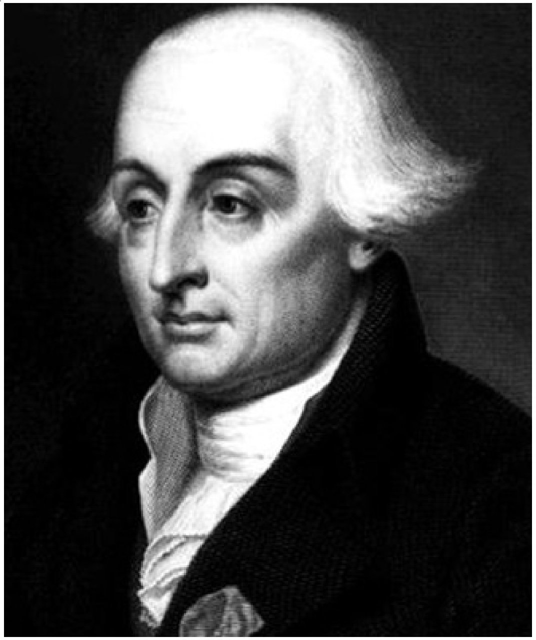
$$\lambda_{observed} = (c-v)T_{observed} = (c-v)\gamma T_{source} = \frac{1-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} cT_{source}$$

$$\frac{\lambda_{observed}}{\lambda_{source}} = \frac{f_{source}}{f_{observed}} = \sqrt{\frac{1-\beta}{1+\beta}} \longrightarrow \sqrt{\frac{1/2\gamma^2}{2}} = \frac{1}{2\gamma}$$

Ex] Fundamental radiation wavelength from undulator, which is a periodic arrangement (λ_u) of many short dipole magnets of alternating polarity.

- Electron sees length contraction of the undulator period: $\lambda_u' = \lambda_u/\gamma$
- The electrons oscillate at a corresponding higher frequency: $\omega = 2\pi c/\lambda_u'$
- The electrons emit radiation just like an oscillation dipole: $P = (e^2/6\pi\epsilon_0 c^3)\dot{\mathbf{v}}^2 \propto \omega^4$
- For a stationary observer looking against the electron beam, the radiation appears strongly blue-shifted: $\lambda_{observed} \approx \lambda_u'/(2\gamma) \approx \lambda_u/(2\gamma^2)$

Classical Mechanics



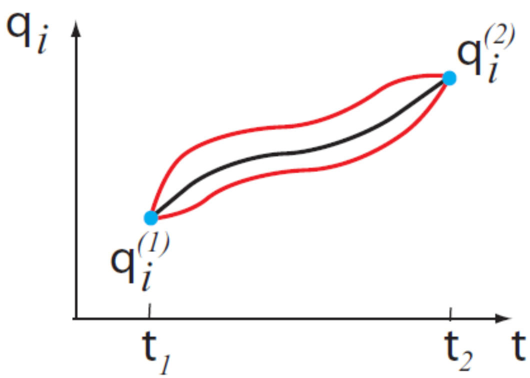
Lagrangian Mechanics

- If we take the nonrelativistic case of a conservative system and **B** = 0, the Lagrange function is defined by the difference between kinetic and potential energy.

$$L = T - V$$

- Hamilton's variational principle states that the motion of the system from one fixed point at time t_1 to another point at time t_2 is such that the time integral of the Lagrangian along the path taken is an extremum (actually, a minimum).

$$\delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)\ dt = \int_{t_1}^{t_2} \delta L(\mathbf{q}, \dot{\mathbf{q}}, t)\ dt = 0$$



- Lagrangian equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- Canonical momenta (or conjugate momenta):

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \equiv p_{c,i}$$ in the FOBP textbook

Hamiltonian Mechanics

- Hamiltonian is constructed from a Lagrangian:

$$H(\mathbf{q},\mathbf{p},t) = \mathbf{p}\cdot\dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}},t)$$

- Hamiltonian equations of motion:

$$\begin{aligned} dH &= \sum_i \left[\left(\frac{\partial H}{\partial q_i}\right) dq_i + \left(\frac{\partial H}{\partial p_i}\right) dp_i \right] + \left(\frac{\partial H}{\partial t}\right) dt \\ &= \sum_i \left[\dot{q}_i dp_i + p_i d\dot{q}_i - \underbrace{\left(\frac{\partial L}{\partial q_i}\right)}_{=\dot{p}_i} dq_i - \underbrace{\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{=p_i} d\dot{q}_i \right] - \left(\frac{\partial L}{\partial t}\right) dt \end{aligned}$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

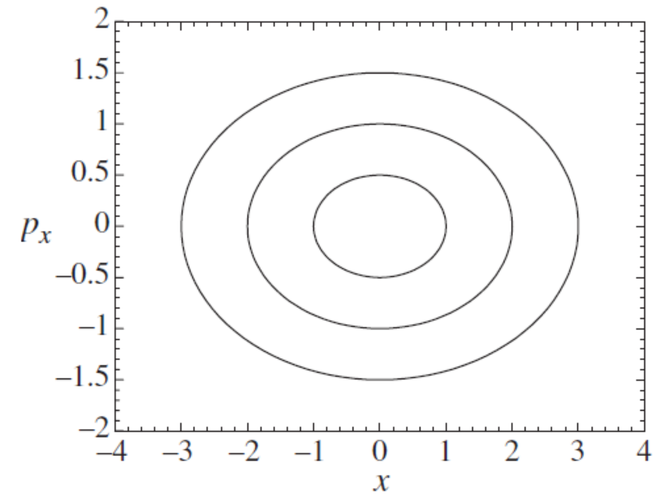
- Conservation of Hamiltonian: if it does not depend explicitly on t, it becomes a conservative system (Hamiltonian systems need not be conservative in general).

$$\frac{dH}{dt} = \sum_i \left[\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i \right] + \frac{\partial H}{\partial t} = \sum_i \left[-\cancel{\dot{p}_i \dot{q}_i} + \cancel{\dot{p}_i \dot{q}_i} \right] + \frac{\partial H}{\partial t}$$

[Example]

- Non-relativistic simple harmonic oscillator:

$$H = \frac{1}{2m}\left[p_x^2 + m^2\omega^2 x^2\right]$$



- Equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -m\omega^2 x \longrightarrow \ddot{x} + \underbrace{\omega^2}_{=k/m} x = 0$$

- Well-known solutions:

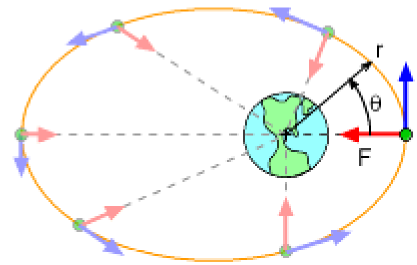
$$x(t) = x_m \sin(\omega t + \theta_0), \quad p_x(t) = m\omega x_m \cos(\omega t + \theta_0)$$

- Area of the phase plane ellipse:

$$\oint p_x dx = \oint p_x \dot{x} dt = \oint (H + L) dt = \oint 2T dt = \oint m\omega^2 x_m^2 \cos^2(\omega t + \theta_0) dt = \frac{1}{2} k x_m^2 \tau = E\tau$$

Total energy

[Example]



- Lagrangian for a central force problem in 2D:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

- Canonical momenta:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

- Hamiltonian:

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)$$

- Equations of motion:

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \quad \frac{dp_\theta}{dt} = -\frac{\partial H}{\partial \theta} = 0 \longrightarrow p_\theta = const. \equiv l = \text{angluar momentum}$$

$$\frac{dr}{dt} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = \frac{l^2}{mr^3} - \frac{\partial V(r)}{\partial r}$$

Generalized Potential in EM Fields

- Lorentz force in terms of potentials:

$$
\begin{aligned}
\mathbf{F} &= \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\
&= q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\right] \\
&= q\left[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} - (\mathbf{v} \cdot \nabla)\mathbf{A} + \nabla(\mathbf{v} \cdot \mathbf{A})\right] \\
&= q\left[-\nabla\left(\phi - \mathbf{v} \cdot \mathbf{A}\right) - \frac{d\mathbf{A}}{dt}\right]
\end{aligned}
$$

  – We used the fact that (x, y, z) and (v_x, v_y, v_z) are independent variables, and

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

- If we recognize:

$$\frac{dA_x}{dt} = \frac{d}{dt}\left[\frac{\partial}{\partial v_x}(\mathbf{v} \cdot \mathbf{A})\right]$$

$$F_x = q\left\{-\frac{\partial}{\partial x}(\phi - \mathbf{v} \cdot \mathbf{A}) - \frac{d}{dt}\left[\frac{\partial}{\partial v_x}(\mathbf{v} \cdot \mathbf{A})\right]\right\} = -\frac{\partial U^*}{\partial x} + \frac{d}{dt}\left[\frac{\partial U^*}{\partial v_x}\right]$$

  – Here, we used the fact scalar potential is independent of velocity.
  – $U^* = q\phi - q\boldsymbol{v} \cdot \boldsymbol{A}$ is the generalized potential used in Lagrangian

Relativistic Dynamics in EM Fields

- Lagrangian with velocity-dependent potentials U*:

$$L = -mc^2\left(1 - v^2/c^2\right)^{1/2} - q\phi + q\mathbf{v}\cdot\mathbf{A}$$

To derive it from the scratch is beyond scope. Just check, $\frac{\partial L}{\partial v_x} = -mc^2 \times \left(-\frac{1}{2}\gamma\right)\left(-\frac{2v_x}{c^2}\right) = \gamma m v_x$

- Canonical momenta: mechanical (kinetic) momenta + vector potential contribution

$$\mathbf{v} = (\dot{x}, \dot{y}, \dot{z}) \quad \text{or} \quad (\dot{r}, r\dot{\theta}, \dot{z})$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Ex] Cartesian coordinates:

$$p_x = \frac{\partial L}{\partial \dot{x}} = p_{mech,x} + qA_x = \gamma m\dot{x} + qA_x$$

Ex] Cylindrical coordinates:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = p_{mech,\theta} + qrA_\theta = \gamma m r^2\dot{\theta} + qrA_\theta$$

Relativistic Dynamics in EM Fields

- Hamiltonian: Using Cartesian coordinates, one can prove

$$
\begin{aligned}
H &= \sum_i p_i \dot{q}_i - L \\
&= \mathbf{v} \cdot (\gamma m \mathbf{v} + q\mathbf{A}) + mc^2\sqrt{1 - v^2/c^2} - q\mathbf{v} \cdot \mathbf{A} + q\phi \\
&= \gamma mc^2 \left(\beta^2 + \frac{1}{\gamma^2}\right) + q\phi = \gamma mc^2 + q\phi \\
&= \sqrt{\gamma^2 m^2 v^2 c^2 + m^2 c^4} + q\phi = \sqrt{p_{mech}^2 c^2 + m^2 c^4} + q\phi
\end{aligned}
$$

Ex] Cartesian coordinates:

$$
\begin{aligned}
H &= c\left[(\mathbf{p} - q\mathbf{A})^2 + m^2c^2\right]^{1/2} + q\phi \\
&= c\left[(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2 + m^2c^2\right]^{1/2} + q\phi
\end{aligned}
$$

Ex] 4-vectors:

$$
\left(\mathbf{p} - q\mathbf{A}, \frac{i(H - q\phi)}{c}\right) \longrightarrow (\mathbf{p} - q\mathbf{A})^2 - \frac{(H - q\phi)^2}{c^2} = -m^2c^2
$$

Ex] Cylindrical coordinates:

$$
H = c\left[(p_r - qA_r)^2 + \left(\frac{p_\theta - qrA_\theta}{r}\right)^2 + (p_z - qA_z)^2 + m^2c^2\right]^{1/2} + q\phi
$$

Canonical Transformation

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- The variation of the action integral between two fixed endpoints:

$$\delta \int_{t_1}^{t_2} L \; dt = \delta \int_{t_1}^{t_2} [\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t)] \; dt = 0$$

- We would like to transform from the old coordinate system $(\mathbf{q}, \mathbf{p})$ to a new system $(\mathbf{Q}, \mathbf{P})$ with a new Hamiltonian $K(\mathbf{Q}, \mathbf{P}, t)$:

$$\delta \int_{t_1}^{t_2} [\mathbf{P} \cdot \dot{\mathbf{Q}} - K(\mathbf{Q}, \mathbf{P}, t)] \; dt = 0$$

- One way for both vibrational integral equalities to be satisfied is to have

$$\lambda[\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p}, t)] = \mathbf{P} \cdot \dot{\mathbf{Q}} - K(\mathbf{Q}, \mathbf{P}, t) + \frac{dF}{dt}$$

- If $\lambda \neq 1$, it is extended canonical transformation. If $\lambda \neq 1$ and $\frac{dF}{dt} = 0$, it is scale transformation. These transformations do not preserve phase space volume

Generating Function

- The function F is in general a function of both the old and new variables as well as the time. We will restrict ourselves to functions that contain half of the old variables and half the new; these are useful for determining the explicit form of the transformation.

Case 1:

$$F = F_1(\mathbf{q}, \mathbf{Q}, t) \qquad p_i = +\frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$$

Case 2:

$$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q}\cdot\mathbf{P} \qquad p_i = +\frac{\partial F_2}{\partial q_i}, \quad Q_i = +\frac{\partial F_2}{\partial P_i}$$

Case 3:

$$F = F_3(\mathbf{Q}, \mathbf{p}, t) + \mathbf{q}\cdot\mathbf{p} \qquad q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

Case 4:

$$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q}\cdot\mathbf{p} - \mathbf{Q}\cdot\mathbf{P} \qquad q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = +\frac{\partial F_4}{\partial P_i}$$

- In all cases, new Hamiltonian and equations of motion become:

$$K = H + \frac{\partial F_i}{\partial t}, \qquad \frac{dQ_i}{dt} = \frac{\partial K}{\partial P_i}, \qquad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q_i}$$

[Example]

- For F_3 we will show

$$\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t) = \mathbf{P}\cdot\dot{\mathbf{Q}} - K(\mathbf{Q},\mathbf{P},t) + \frac{dF}{dt}$$

- Proof:

$$F = F_3(\mathbf{Q},\mathbf{p},t) + \mathbf{q}\cdot\mathbf{p}$$

$$\begin{aligned}
\frac{dF}{dt} &= \frac{\partial F_3}{\partial t} + \sum_i \left(\underbrace{\frac{\partial F_3}{\partial Q_i}}_{=-P_i} \dot{Q}_i + \underbrace{\frac{\partial F_3}{\partial p_i}}_{=-q_i} \dot{p}_i + +q_i\dot{p}_i + p_i\dot{q}_i \right) \\
&= \frac{\partial F_3}{\partial t} + \sum_i \left(-P_i\dot{Q}_i - \cancel{q_i\dot{p}_i} + \cancel{q_i\dot{p}_i} + p_i\dot{q}_i \right) \\
&= \frac{\partial F_3}{\partial t} - \mathbf{P}\cdot\dot{\mathbf{Q}} + \mathbf{p}\cdot\dot{\mathbf{q}}
\end{aligned}$$

Therefore

$$K = H + \frac{\partial F_3}{\partial t} \longrightarrow \mathbf{p}\cdot\dot{\mathbf{q}} - H = \mathbf{P}\cdot\dot{\mathbf{Q}} - K + \frac{dF}{dt}$$

[Example]

- For non-relativistic simple harmonic oscillator:

$$x(t) = x_m \sin(\omega t + \theta_0), \quad p_x(t) = m\omega x_m \cos(\omega t + \theta_0)$$

- Let's use generation function of type 1: $F_1(q, Q, t) = F(x, \theta), \quad \theta = \omega t + \theta_0$

$$p_x = \frac{\partial}{\partial x} F(x,\theta), \quad P_x \equiv J = -\frac{\partial}{\partial \theta} F(x,\theta), \quad H' = H + \cancel{\frac{\partial}{\partial t} F(x,\theta)}$$

$$\longrightarrow F(x,\theta) = \int p_x dx = \int \left(m\omega \frac{x}{\sin\theta} \cos\theta \right) dx = \frac{1}{2} m\omega x^2 \cot(\theta)$$

$$\longrightarrow J = -\frac{\partial}{\partial \theta} \left(\frac{1}{2} m\omega x^2 \cot(\theta) \right) = \frac{1}{2} m\omega x^2 \frac{1}{\sin^2(\theta)} = \frac{1}{2} m\omega x_m^2 = U/\omega$$

$$\longrightarrow H' = H = U = J\omega$$

- Action–angle variables: J and θ (new canonical variables instead of x, p_x)

- The action (J) is also generally known to be an adiabatic invariant:

$$J = \frac{1}{2\pi} \oint p_x dx = \text{remains constant when the parameters of an oscillatory system are changed slowly}$$

Change the Role of Time Coordinates

- Provided that the reference particle moves without backtracking, or some particle coordinate q_j increases in time, we can change the role of that coordinate and time.

$$\int_{t_1}^{t_2} [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]\, dt = \int_{t_1}^{t_2} [\mathbf{p}\cdot d\mathbf{q} - H(\mathbf{q},\mathbf{p},t)dt]$$

$$\mathbf{p}\cdot d\mathbf{q} - H dt = \sum_i p_i dq_i + (-H)dt = \left(\sum_{i\neq j} p_i dq_i + (-H)dt\right) - (-p_j)dq_j$$

$$H \longleftrightarrow -p_j$$
$$t \longleftrightarrow q_j$$

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \longleftrightarrow \frac{dt}{dq_j} = \frac{\partial(-p_j)}{\partial(-H)}, \quad \frac{d(-H)}{dq_j} = -\frac{\partial(-p_j)}{\partial t}$$

Paraxial Approximation

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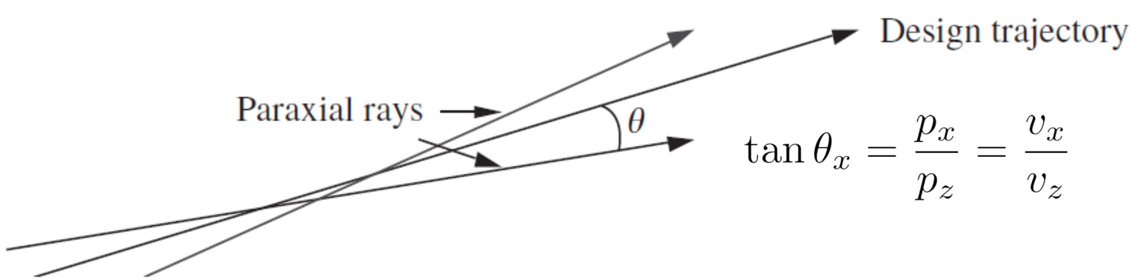
- Design trajectory (Design orbit for closed system): Ideally preferred trajectory through the system (a locally Cartesian or curvilinear in circular machines)
- Paraxial rays: Vector representations of the local trajectory, which have an angle with respect to a design trajectory that is much smaller than unity.



$$\tan\theta_x = \frac{p_x}{p_z} = \frac{v_x}{v_z}$$

- Distance along the design trajectory (e.g., z) as the independent variable rather that time t: Because in an optic system (for charged particles or photons) the forces encountered are always specified in space not in time.
- Paraxial approximation:

$$p_{x,y} \ll p_z \simeq |\mathbf{p}|$$

$$x' = \frac{dx}{dz} = \tan(\theta_x) \simeq \theta_x \simeq \sin(\theta_x) \ll 1$$

$$\ddot{x} + \omega^2 x = 0 \longrightarrow x'' + k^2 x = 0, \quad k = \omega/v_z \longrightarrow x_m k \sim \frac{x_m}{\lambda} \ll 1$$

Distribution Function

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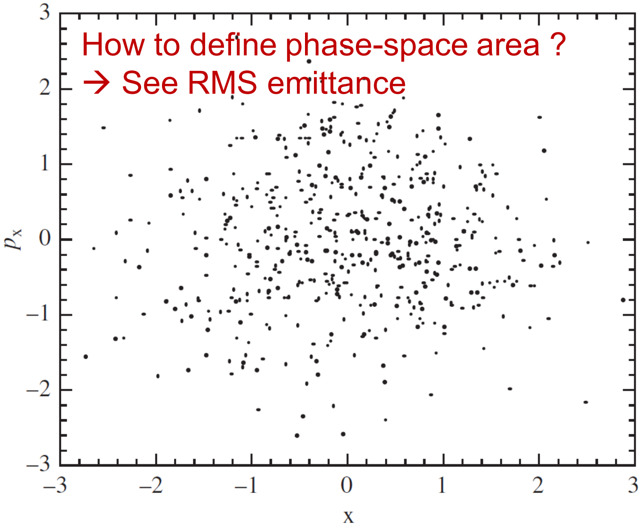
- Distribution function:

Discrete (real world)



Continuous (mathematical approximation)

$$\longrightarrow f(\mathbf{x}, \mathbf{p}, t) d^3\mathbf{x} d^3\mathbf{p} =$$

The number of particles found in a differential volume in the neighborhood of a phase space location ***x, p*** at a time *t*

- With a smooth phase space distribution, the charge and current distributions associated with such a distribution are also continuous and smooth.
- The fields derived from the smooth charge/current densities may be termed macroscopic. Deviations from these approximate fields (near an individual particle) may be termed microscopic.

Liouville's Theorem

- Total time derivative of the distribution function:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \nabla f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f$$

- From continuity in phase-space:

$$0 = \frac{\partial f}{\partial t} + \nabla \cdot (\dot{\mathbf{x}} f) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}} f)$$

- If the forces are derivable from a Hamiltonian:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{dx_i}{dt}\frac{\partial f}{\partial x_i} + \frac{dp_i}{dt}\frac{\partial f}{\partial p_i} \right)$$

$$= -\sum_i \left(\frac{\partial \dot{x}_i}{\partial x_i} f + \frac{\partial \dot{p}_i}{\partial p_i} f \right) = -\sum_i f \left[\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial x_i}\right) \right] = 0$$

- In other words, when no dissipative forces (i.e., ignoring degree of freedom), no particle lost or created, and no small-impact-parameter binary Coulomb collisions between particles:

$$\frac{df}{dt} = 0$$

Continuity: $0 = \frac{\partial f}{\partial t} + \nabla \cdot (\dot{\mathbf{x}} f) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}} f) \longrightarrow \nabla \cdot (\dot{\mathbf{x}}) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}}) = 0$:Imcompressibility

Some Comments on Liouville's Theorem

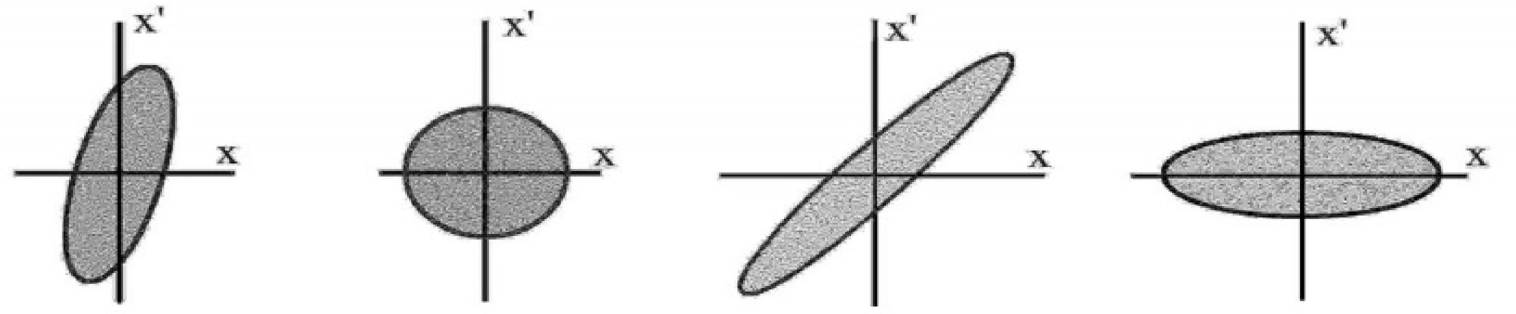
- Liouville's theorem states that the phase space density encountered as one travels with a particle in a Hamiltonian system is conserved.
  - The density of any volume of phase space whose boundary follows the Hamiltonian equations is constant.
  - The volume occupied by particles in phase space (=emittance) is conserved (shape may change).

- Liouville's theorem is valid not only for the time-independent Hamiltonian case, but also for the time-dependent Hamiltonian case.
- Liouville's theorem is valid for both equilibrium and non-equilibrium systems.
- Liouville's theorem is valid for both linear and non-linear systems.

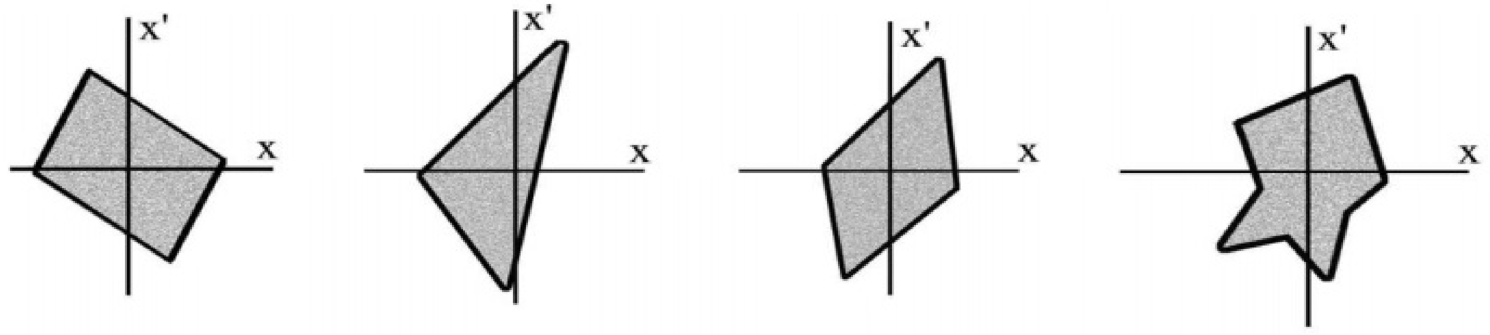
- Liouville's theorem does not imply that the density is uniform throughout phase space.
- Liouville's theorem only holds in the limit that the particles are infinitely close together. Equivalently, Liouville's theorem does not hold for any ensemble that consists of a finite number of particles.
- Liouville's theorem holds even in the presence of space-charge and wake-fields, but not with microscopic binary collisions.

Emittance

The phase space area/volume (emittance) occupied by a particle beam is an invariant. (Incompressible flow)



Phase space ‘footprints’ of the same beam

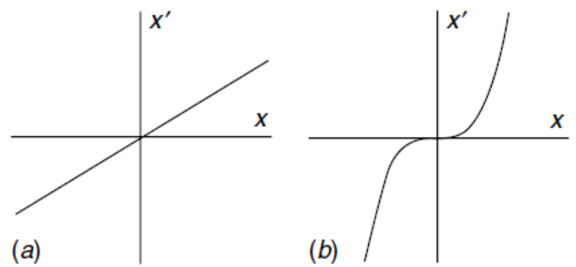


RMS Emittance

- In the case of a real beam with a finite number of particles (N), an RMS emittance can be defined for an effective phase-space area (or volume).

$$\epsilon_{\rm rms} = \sqrt{\langle x^2\rangle \langle p_x^2\rangle - \langle x p_x\rangle^2}$$

→ Depends not only on the true area occupied by the beam in phase space, but also on the distortions produced by nonlinear forces.
→ Hamiltonian flow: Phase-space conservation + Entropy growth: Filamentation

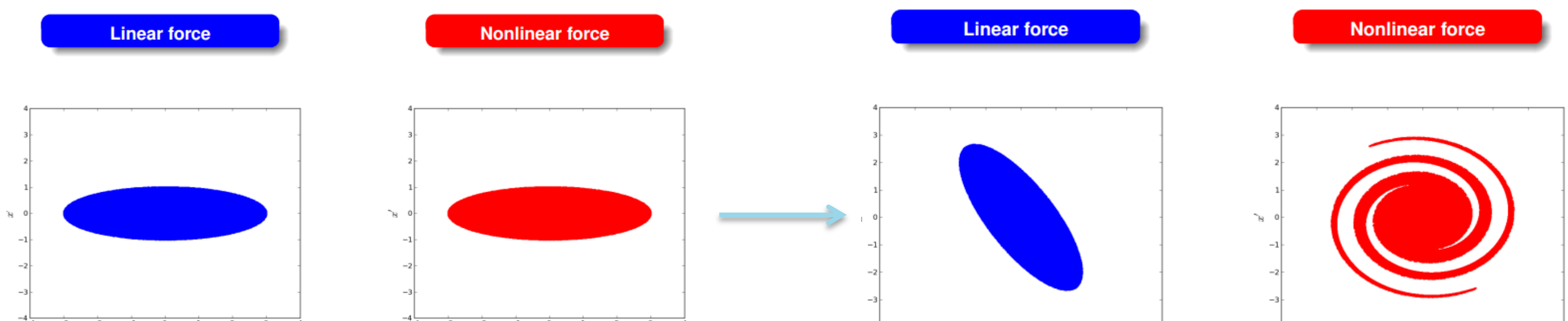


Phase-space area = 0
RMS emittance = 0

Phase-space area = 0
RMS emittance > 0

- However, when nonlinear forces act on the system, e.g. nonlinear magnetic fields, space charge force, the RMS emittance is not conserved.

Filamentation
→ Dilution of phase space density



For finite N, average distance of the particles in one spiral equals the distance between two adjacent spirals.

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