CSE515 Advanced Algorithms
Notes on Lecture 27: Tail Inequalities

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First question: Suppose the average running time of your randomized algorithm is 1h. What can you say about the probability that it runs for more than 10h?

Answer. The probability is at most 1/10. Proof: If it were more than 1/10, then the average running time would be more than 10h/10=1h, a contradiction.

Theorem 1. *Let T be a random BST over n nodes $a_1, \dots, a_n$. Then for each i,*

$$\Pr[\text{depth}(a_i) \geqslant 32 \ln n] \leqslant \frac{1}{n^4}$$

Proof. In the previous lecture, we defined a *central* node in a BST to be a node v such that the size of each of the subtree rooted at v is at most $3|T_v|/4$, where $|T_v|$ is the size (the number of nodes) of T_v. Then a node in a random BST is central with probability 1/2.

Let $u = a_i$ be the node whose depth we want to estimate, and let P_u be the path from the root to u. We saw in last lecture that there are at most $\log_{4/3} n$ central nodes along P_u, since the size of the subtree rooted at the current node decreases by a factor at least 4/3 at each central node.

We introduce the random variable X_i which is equal to 1 if the ith node of P_u is central, and $X_i = 0$ otherwise. If i is more than the length $|P_u|$ of the path P_u, then we set X_i to be equal to a random bit, hence in all cases we have $\Pr(X_i = 1) = 1/2$, and these probabilities are independent.

We now consider $X = X_1 + \cdots + X_{32 \ln n}$. Suppose $X \geqslant 4 \ln n$. As $\ln(4/3) = 0.28$, then $\log_{4/3} n = \ln n / \ln(4/3) \leqslant 4 \ln n$, hence $X \geqslant \log_{4/3} n$. In this case, there are $\log_{4/3} n$ central nodes along the first $32 \ln n$ nodes of P_u, so the path has length at most $32 \ln n$. So we want to show that the other case, $X \leqslant 4 \ln n$, happens with a small probability.

So we now estimate $\Pr[X \leqslant 4 \ln n]$ by applying the Chernoff bound for lower tail. Here $\mu = 16 \ln n$ so $\delta = 3/4$. It follows that

$$\begin{aligned}
\Pr[\text{depth}(u \geqslant 32 \ln n)] &\leqslant \Pr[X \leqslant 4 \ln n] \leqslant e^{-\frac{\mu\delta^2}{2}} \\
&= e^{-\frac{16(\ln n)(3/4)^2}{2}} = n^{-8\times 9/16} \\
&\leqslant n^{-64/16} = \frac{1}{n^4}
\end{aligned}$$

□