Advanced Algorithms
Lecture 11
The Simplex Algorithm II

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Introduction

- This is the second part of the lecture on the *simplex algorithm*.
- Reference: Chapter 29.3 of the textbook Introduction to Algorithms by Cormen, Leiserson, Rivest and Stein.

Proof of Correctness

Definition

We say that two slack forms are *equivalent* if they have the same set of feasible solutions.

Lemma

All the slack forms produced by the simplex algorithm are equivalent.

Proof.

At each pivot, we first move x_e to the LHS, obtaining an equivalent equation. Then this equation multiplied by a constant is added to each other equality constraint. As in Gaussian elimination, it produces an equivalent system of equations. □

Proof of Correctness

Lemma

For a given LP, and for a given choice of basic variables, the simplex algorithm cannot produce two different slack forms.

Proof.

Done in class. Lemma 29.3 and 29.4 page 876 in the textbook. □

It follows that:

Corollary

If cycling does not occur, then the simplex algorithm terminates in at most $\binom{n+m}{n}$ *steps.*

General Case

Simplex(A, b, c)

$(N, B, A, b, c, \nu) \leftarrow$ Initialize-Simplex(A, b, c)
while $\exists j : c_j > 0$ **do**
  Choose e such that $c_e > 0$
  for each $i \in B$ **do**
    if $a_{ie} > 0$ **then** $\Delta_i \leftarrow b_i / a_{ie}$
    else $\Delta_i \leftarrow \infty$
  Choose ℓ that minimizes Δ_ℓ
  if $\Delta_\ell = \infty$ **then** **return** unbounded
  else $(N, B, A, b, c, \nu) \leftarrow$ Pivot($N, B, A, b, c, \nu, \ell, e$)
for $i \leftarrow 1, n$ **do**
  if $i \in B$ **then** $\bar{x}_i \leftarrow b_i$
  else $\bar{x}_i \leftarrow 0$
return $(\bar{x}_1, \ldots, \bar{x}_n)$

Proof of Correctness

In this slide, we assume that the `Initialize-Simplex` procedure from the previous slide returns a slack form whose basic solution is feasible. (This procedure is described in the textbook, Section 29.5.)

Lemma

If the simplex algorithm returns `unbounded`, *then the linear program is unbounded.*

Lemma

If the simplex algorithm returns $(\bar{x}_1, \ldots, \bar{x}_n)$, *then it is an optimal solution.*

Theorem

If cycling does not occur, then the simplex algorithm returns a correct answer after at most $\binom{n+m}{n}$ *iterations.*

The Initial Basic Feasible Solution

- Consider the following LP:

$$\begin{array}{lrcrl} \text{maximize} & 2x_1 & - & x_2 & \\ \text{subject to} & 2x_1 & - & x_2 & \leqslant 2 \\ & x_1 & - & 5x_2 & \leqslant -4 \\ & & & x_1, x_2 & \geqslant 0 \end{array}$$

- Suppose we want to solve it with the simplex algorithm.
- After converting into slack form:

$$\begin{array}{rcrcrcr} z & = & & & 2x_1 & - & x_2 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 \end{array}$$

- What is the problem?
  - The basic solution is not feasible.

Auxiliary Linear Program

- Let L be a LP in standard form:

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leqslant b_i, & i = 1, \dots, m, \\ & x_j \geqslant 0, & j = 1, \dots, n. \end{array}$$

- The *auxiliary linear program* L_{aux} is:

$$\begin{array}{lll} \text{maximize} & -x_0 & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leqslant b_i, & i = 1, \dots, m, \\ & x_j \geqslant 0, & j = 0, \dots, n. \end{array}$$

Auxiliary Linear Program

Proposition

The linear program L is feasible if and only if the optimal objective value of L_{aux} is 0.

Proof: Done in class.

Example

- The auxiliary LP for the LP in Slide 8 is:

$$\begin{array}{lrcrcrl} \text{maximize} & & & & - & x_0 & \\ \text{subject to} & 2x_1 & - & x_2 & - & x_0 & \leqslant 2 \\ & x_1 & - & 5x_2 & - & x_0 & \leqslant -4 \\ & & & & x_1, x_2, & x_0 & \geqslant 0 \end{array}$$

- We solve this LP using the simplex algorithm.
- The first slack form is:

$$\begin{array}{rcrcrcrcr} z & = & & & & & & - & x_0 \\ x_3 & = & 2 & - & 2x_1 & + & x_2 & + & x_0 \\ x_4 & = & -4 & - & x_1 & + & 5x_2 & + & x_0 \end{array}$$

- The basic solution is not feasible.
- We choose x_0 and x_4 as the entering and leaving variables, respectively.

Example

- The new slack form is:

$$\begin{array}{rcrcrcrcr} z & = & -4 & - & x_1 & + & 5x_2 & - & x_4 \\ x_0 & = & 4 & + & x_1 & - & 5x_2 & + & x_4 \\ x_3 & = & 6 & - & x_1 & - & 4x_2 & + & x_4 \end{array}$$

- The basic solution is now feasible. (It will always be the case.)
- We now run the simplex algorithm until we find an optimal solution.
- We pick $x_e = x_2$ and $x_\ell = x_0$, and thus:

$$\begin{array}{rcrcrcrcr} z & = & & - & x_0 & & & & \\ x_2 & = & \frac{4}{5} & - & \frac{1}{5}x_0 & + & \frac{1}{5}x_1 & + & \frac{1}{5}x_4 \\ x_3 & = & \frac{14}{5} & + & \frac{4}{5}x_0 & - & \frac{9}{5}x_1 & + & \frac{1}{5}x_4 \end{array}$$

- The optimal value for L_{aux} is 0, so the original LP is feasible.

Example

- As $x_0 = 0$, we remove it from the slack form:

$$
\begin{array}{rcrcrcr}
z & = & ? & & & & \\
x_2 & = & \frac{4}{5} & + & \frac{1}{5}x_1 & + & \frac{1}{5}x_4 \\
x_3 & = & \frac{14}{5} & - & \frac{9}{5}x_1 & + & \frac{1}{5}x_4
\end{array}
$$

- We restore the original objective function

$$
\begin{aligned}
z &= 2x_1 - x_2 \\
&= 2x_1 - \left(\frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4\right)
\end{aligned}
$$

Example

- We obtain the following slack form, equivalent to the original LP:

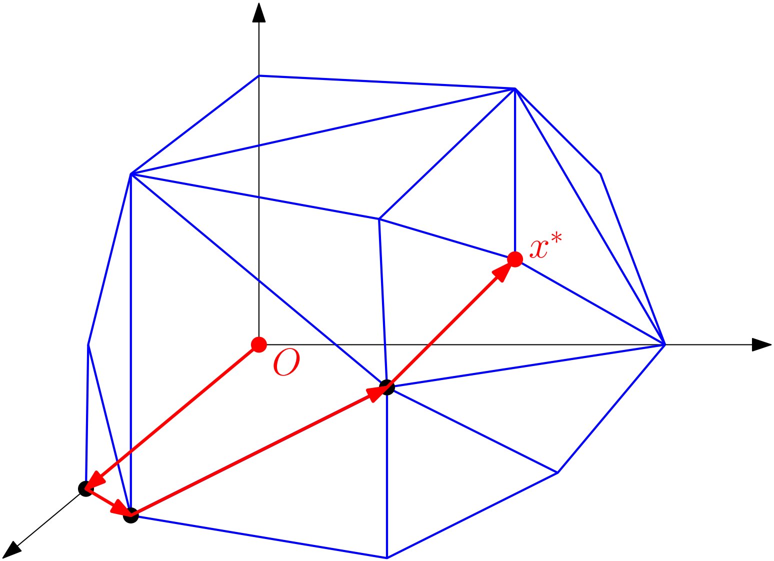
$$
\begin{aligned}
z &= -\frac{4}{5} + \frac{9}{5}x_1 - \frac{1}{5}x_4 \\
x_2 &= \frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\
x_3 &= \frac{14}{5} - \frac{9}{5}x_1 + \frac{1}{5}x_4
\end{aligned}
$$

- This slack form has a feasible basic solution, so this completes the execution of Initialize-Simplex.

General Case

- Construct L_{aux}
- Make a first pivot with $e = 0$ and $\ell = k$ such that b_k is minimum
- The basic solution of L_{aux} is now feasible
- Solve L_{aux} with the simplex algorithm.
- If $\bar{x}_0 \neq 0$ in the basic solution, then the LP is unfeasible.
- Othwerwise (if $\bar{x}_0 = 0$)
  - If x_0 is basic, make it nonbasic by performing one pivot.
  - Take out x_0 from the slack form. Now the basic solution is a feasible solution to the original program.
- More details can be found in textbook Section 29.5

Geometry



Geometry

- The simplex algorithm moves from one vertex of the feasible region to a neighboring vertex.
- At each move, the objective function does not decrease.
- For instance, it starts from the vertex $(x_1, \dots, x_n) = (0, \dots, 0)$, which is the initial basic solution restricted to $(x_1, \dots, x_n)$.
- At each step, the n nonbasic variables N give a set of n variables that are set to 0 in the basic solution.
- It corresponds to n of the constraints of the original LP being satisfied.
- In other words, the current basic solution is at the intersection of n hyperplanes bounding the feasible region.
- So it is a vertex of the feasible region.