

Lecture 1 'Single' Charged Particle Motion in Static Fields (Ref: Ch. 2 of UP-ALP, Ch. 2 of FOBP)

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[Basics] Main Results of Special Relativity

• Relativistic parameters: Don't be confused with Twiss parameters.

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1 + \frac{1}{2}\beta^2, \quad \beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}$$



Energy is increased, but velocity is nearly constant

The total energy, mechanical momentum, and kinetic energy of a rest mass m:

$$E = \gamma mc^2, \ p = \gamma m\beta c, \ T = (\gamma - 1)mc^2 \longrightarrow \frac{1}{2}mv^2 \ (v \ll c)$$

• The relation between total energy and momentum in the absence of EM fields:

$$E = \sqrt{p^2 c^2 + (mc^2)^2}, \text{ or } \gamma^2 = (\beta \gamma)^2 + 1$$



[Basics] Energy and Mass Units

 To describe the energy of individual particles, we use the eV, the energy that a unit charge

 $e = 1.6 \times 10^{-19}$ Coulomb

gains when it falls through a potential, $\Delta \phi = 1$ volt.

 $1 \text{ eV} = 1.6 \times 10^{-19} \text{ Joule}$

• We can use Einstein's relation to convert rest mass to energy units.

$$E_{rest} = mc^2$$

• For electrons,

$$E_{rest} = (9.11 \times 10^{-31} \text{ kg}) \times (3 \times 10^8 \text{ m/s})^2 = 0.511 \text{ MeV}$$

• For protons,

$$E_{rest} = (1.67 \times 10^{-27} \text{ kg}) \times (3 \times 10^8 \text{ m/s})^2 = 938 \text{ MeV}$$



1.1 Motion in a uniform magnetic field

• Motion of charged particle in EM fields:

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \mathbf{F}, \quad \frac{d\mathcal{E}}{dt} = \mathbf{F} \cdot \mathbf{v}$$

- where the momentum $m{p}$ and energy $m{\mathcal{E}}$ of the particle are given by $m{p}=m_0\gammam{v},~~\mathcal{E}=m_0\gamma c^2$
- and m_0 is particle rest mass, and γ is the relativistic mass factor.
- In a uniform magnetic field: Conservation of energy

$$m_0 \frac{d\gamma \mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}, \quad \frac{d\mathcal{E}}{dt} = 0 \longrightarrow \mathcal{E} = const., \quad p = \sqrt{p_\perp^2 + p_z^2} = const.$$

• Equations of motion: $\mathbf{B} = B\hat{z}$

$$m_0 \gamma \dot{v}_x = q v_y B$$
 and $m_0 \gamma \dot{v}_y = -q v_x B$, $\dot{v}_z = 0$

$$\ddot{v}_x = \frac{qB}{m_0\gamma}\dot{v}_y = -\left(\frac{qB}{m_0\gamma}\right)^2 v_x$$

• Cyclotron frequency:

$$\omega_c = \left| \frac{qB}{\gamma m_0} \right|$$



phase

١Ζ

B₀

1.1 Motion in a uniform magnetic field (cont'd)

• General solutions:

$$v_x = -v_m \cos(\omega_c t + \phi), \quad v_y = \pm v_m \sin(\omega_c t + \phi)$$
$$x = R \sin(\omega_c t + \phi) + x_0, \quad y = \pm R \cos(\omega_c t + \phi) + y_0$$

• Meaning of the parameters:

$$v_x^2 + v_y^2 = v_m^2 = v_\perp^2, \quad R = v_m/\omega_c = \rho, \quad (x - x_0)^2 + (y - y_0)^2 = R^2$$

• Physical meaning: Balance of radial force (Lorentz force, F_L) and centripetal force

$$\frac{\gamma m_0 v_{\perp}^2}{R} = q v_{\perp} B_0 \quad \text{or} \quad p_{\perp} = q B_0 R$$

• Magnetic rigidity [T m]:

$$B_0 R = \frac{p_\perp}{q}, \text{ or } B\rho = \frac{p}{q}$$

• Pitch angle: Associated with helical motion

$$heta_p = an^{-1}(p_z/p_\perp)$$
 For q >0



[Note] Circular Accelerator VS Focusing Solenoid

- Circular accelerator:
 - Very small pitch angle
 - Indeed, $p_z \approx 0$
 - Circular design orbit



- Very large pitch angle
- Indeed, $p_z \gg p_\perp$
- Straight design orbit



→Be careful ! Later we will reassign x, y, and z coordinates. →In fact, B fields are applied locally around the particle orbits.



1.2 Circular accelerator

- We analyze the charged particle dynamics near the design orbit. The design orbit is specified by a certain radius of curvature (*R*) and a certain momentum ($p_0 = qB_0R$)
- A new locally defined right-handed coordinate system:



• Equation of motion in this new coordinate system: Homework (Problem 2.1 of FOBP)

$$\frac{dp_{\rho}}{dt} = \frac{\gamma m_0 v_{\phi}^2}{\rho} - q v_{\phi} B_0$$
(2.9)

• The azimuthal velocity and radial momentum:

Reference $v_{\phi} = \rho \dot{\phi} \neq \dot{s} \equiv v_0$, and $p_{\rho} = \gamma m_0 \dot{\rho} = \gamma m_0 \dot{x} = p_x$ $v_{\phi} \approx v_0$ Individual particle's velocity Moses Chung | Lecture 1 Single Particle Motion



1.2 Circular accelerator (cont'd)

• Linearization of Eq. (2.9) by assuming $x \ll R$: Lowest order Taylor series expansion about the design orbit equilibrium ($p_x = p_\rho = 0$ at $\rho = R$),

$$\frac{dp_x}{dt} = \frac{\gamma m_0 v_0^2}{R(1+x/R)} - qv_0 B_0 \simeq \frac{\gamma m_0 v_0^2}{R} (1-x/R+\cdots) - qv_0 B_0$$
$$\simeq -\frac{\gamma m_0 v_0^2}{R^2} x + \frac{\gamma m_0 v_0^2}{R} - qv_0 B_0$$

• Using the definition of the design radius and cyclotron frequency we have:

$$\frac{d^2x}{dt^2} + \omega_c^2 x = 0$$

• Using *s* as the independent variable we have:

$$(\cdots)' \equiv \frac{d}{ds} = \frac{d}{v_0 dt}$$
$$x'' + \left(\frac{1}{R}\right)^2 x = 0$$



 Simple harmonic oscillations about the design orbit for particles having the same momentum (p₀ = qB₀R): We call it Betatron oscillations → Basis of phase advance and tunes.



[Note] Dispersion

• Displacement of an arbitrary particle from the design orbit due to deviation from the design momentum:

$$\delta p \equiv p - p_0$$

• An analysis which treats the particle dynamics only in a first-order Taylor series in both betatron (i.e., angle and offset) errors and momentum errors is by assumption a description which is additive in these quantities:



- The coefficient η_x (or D_x) is termed the dispersion, and is generally a detailed function of the magnetic field profile with variation in *s*.
- In case of a uniform magnetic field, the dispersion is constant at the design momentum:

$$\eta_x = \frac{\partial x}{\partial \delta_p} = \frac{\partial x}{\partial \left[\frac{\delta p}{p_0}\right]} = p_0 \frac{\partial x}{\partial (p - p_0)} = p_0 \frac{\partial R}{\partial p} = R(p_0)$$

$$\swarrow$$

$$R(p) = p/qB_0 \qquad \text{R is a function of } p$$



1.3 Focusing in solenoids

- The key to understanding the motion of a charged particle in a focusing solenoid is to recognize how the angular momentum, which drives this helical motion, arises.
- We need to ask what happens when the charged particle moves from a region where the magnetic field vanishes to one where it is uniform.



First order Taylor expansion

$$\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \qquad \mathbf{B} = -\frac{1}{2}B'_z(z)\left(x\hat{x} + y\hat{y}\right) + B_z(z)\hat{z} = -\frac{1}{2}B'_z(z)\rho\hat{\rho} + B_z(z)\hat{z}$$

$$\simeq 2\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z}\left(\text{or }\frac{\partial B_\rho}{\partial \rho} + \frac{B_\rho}{\rho} + \frac{\partial B_z}{\partial z}\right) \qquad B'_z(z) \equiv \left(\frac{\partial B_z}{\partial z}\right)_{(0,0)}$$

$$= 0$$

$$\longrightarrow \frac{\partial B_x}{\partial x} = -\frac{1}{2}\frac{\partial B_z}{\partial z} = -\frac{1}{2}B'_z \qquad \mathbf{A} = \frac{1}{2}B_z(z)\left(x\hat{y} - y\hat{x}\right) = \frac{1}{2}B_z(z)\rho\hat{\theta}$$

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1.3 Focusing in solenoids (cont'd)

• Transverse momentum kick imparted to a charged particle as it passes through the fringe field region: Kick is $-\hat{\theta}$ direction

$$\Delta p_{\perp} \simeq q \int_{t_1}^{t_2} v_z B_{\rho} dt = q \int_{z_1}^{z_2} B_{\rho} dz = -q \frac{\rho_0}{2} \int_{z_1}^{z_2} \left(\frac{\partial B_z}{\partial z}\right)_{\rho=0} dz = -q \frac{\rho_0}{2} \begin{bmatrix} B_z(z_2) - B_z(z_1) \end{bmatrix} = -q \frac{\rho_0}{2} B_0$$

$$B_0 \qquad 0$$
Rotation is CW for $q > 0$

 Rotation in the axisymmetric system due to conservation of canonical angular momentum or Busch's theorem:

$$p_{\theta} = \gamma m \rho^{2} \dot{\theta} + q \rho A_{\theta} = const.$$

$$\gamma m \rho^{2} \dot{\theta} + \frac{q}{2\pi} \Phi = const.$$

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \int \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l} = 2\pi \rho A_{\theta}$$

$$\Phi = 0 \text{ outside the solenoid, but } \Phi \neq 0 \text{ inside.}$$

 A charged particle with no initial transverse motion displays helical motion inside of the solenoid, with radius of curvature such that the particle orbit passes through the axis.

$$R = \frac{|\Delta p_{\perp}|}{qB_0} = \frac{\rho_0}{2} \qquad \qquad const. = \gamma m \rho_0^2 \dot{\theta} + \frac{q}{2\pi} \not{\Phi} = 0 = \gamma m \rho^2 \dot{\theta} + q\rho A_{\theta} \longrightarrow \rho = 0$$
Focusing toward axis



[Note] Equations of motion

 \mathbf{F}

• Equations of motion for constant γ :

$$= q\mathbf{v} \times \mathbf{B}$$

$$= q(\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}) \times \left[-\frac{1}{2}B'_{z}(x\hat{x} + y\hat{y}) + B_{z}\hat{z} \right]$$

$$= qB_{z}(\dot{y}\hat{x} - \dot{x}\hat{y}) + \frac{q\dot{z}B'_{z}}{2}(y\hat{x} - x\hat{y}) \qquad i \left(\frac{\partial B_{z}}{\partial z} \right)_{(0,0)} \approx \frac{dB_{z}}{dt}$$

$$= \underbrace{qB_{z}(\dot{y}\hat{x} - \dot{x}\hat{y})}_{\text{contributal forms}} + \frac{q\dot{B}_{z}}{2}(y\hat{x} - x\hat{y}) \qquad i \left(\frac{\partial B_{z}}{\partial z} \right)_{(0,0)} \approx \frac{dB_{z}}{dt}$$

centripetal force

$$\gamma m_0 \frac{d^2 x}{dt^2} - q B_z \frac{dy}{dt} - \frac{q}{2} \frac{dB_z}{dt} y = 0$$

$$\gamma m_0 \frac{d^2 y}{dt^2} + q B_z \frac{dx}{dt} + \frac{q}{2} \frac{dB_z}{dt} x = 0$$

• Introducing (normalized) Larmor frequency and applying paraxial approximation:

$$\omega_{L} = \frac{\omega_{c}}{2} = \frac{qB_{0}}{2\gamma m_{0}} = -\frac{d\theta_{L}}{dt}$$

$$\sum_{\substack{\text{Different sign convention from the FOBP \\ \text{Rotation is CW for } q > 0}} \Omega_{L} = k_{L} = \frac{\omega_{L}}{v_{z}} = \frac{qB_{0}}{2p_{z}} \simeq \frac{qB_{0}}{2p}$$

$$\frac{d^{2}x}{dt^{2}} - 2\omega_{L}\frac{dy}{dt} - \frac{d\omega_{L}}{dt}y = 0$$

$$\frac{d^{2}y}{dt^{2}} + 2\omega_{L}\frac{dx}{dt} + \frac{d\omega_{L}}{dt}x = 0$$

$$\frac{d^{2}y}{dt^{2}} + 2\omega_{L}\frac{dx}{dt} + \frac{d\omega_{L}}{dt}x = 0$$

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[Note] Larmor frame

 Introducing Larmor frame (the frame that rotates about z-axis with normalized Larmor frequency), in which the transverse orbits in the rotating frame are related to the orbits in the laboratory frame by

$$\left(\begin{array}{c} x_L\\ y_L\end{array}\right) = \left(\begin{array}{c} \cos\theta_L & \sin\theta_L\\ -\sin\theta_L & \cos\theta_L\end{array}\right) \left(\begin{array}{c} x\\ y\end{array}\right)$$



• By direct substitutions, one can show that



[Example] Larmor frame

• Particle orbits become considerably simpler by introducing Larmor frame:





[Example] Larmor frame

• If the particle begins its trajectory offset in x ($x = x_0$), but not in y, and with no transverse momentum before the magnetic field region:



$$\theta_c = 2\theta_L$$

 $\left|\frac{d\theta_c}{dt}\right| = |\omega_c| = \left|2\frac{d\theta_L}{dt}\right| \equiv 2|\omega_L|$

$$x_L = 2 \times \frac{x_0}{2} \cos \theta_L$$
$$= x_0 \cos(\omega_L t)$$
$$y_L = 0$$

 \rightarrow They are solutions of the following simple harmonic oscillators:

$$\ddot{x}_L + \omega_L^2 x_L = 0, \quad \ddot{y}_L + \omega_L^2 y_L = 0$$

[Example] Particle motion through solenoids

- 1. Focusing in one or more Larmor rotation in a uniform long solenoid (as in an image intensifier, i.e., one-to-one mapping).
- 2. The trapping of particles along field lines.

- 3. Focusing from point to point by a thin solenoid.
- Focusing from point to point by a thin

0.00



 $T_c = \frac{2\pi}{\omega_c}, \quad T = \frac{l}{v_z}$







 $\frac{T}{T_c} \gg 1$



1.4 Motion in a uniform electric field

• For uniform electric field $\boldsymbol{E} = E_0 \hat{z}$, with $\boldsymbol{B} = 0$:

•

•

•

$$\frac{dp_z}{dt} = qE_0, \quad \frac{d\mathbf{p}_\perp}{dt} = 0$$

$$E_0\hat{z} = -\frac{\partial\phi_e}{\partial z}\hat{z}$$
In terms of the potential energy, the Hamiltonian (total canonical energy) is given by
$$H = \gamma m_0 c^2 + q\phi_e = \gamma m_0 c^2 - qE_0 z$$
Because the Hamiltonian is independent of time, it is a constant of motion:
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$$

$$const. = H|_{z=0} = \gamma(0)m_0c^2 = \gamma(z)m_0c^2 - qE_0 z$$

$$\gamma(z) = \frac{H|_{z=0}}{m_0c^2} + \frac{qE_0}{m_0c^2}z = \gamma|_{z=0} + \frac{qE_0}{m_0c^2}z$$
Linear increase in mechanical energy $U = \gamma m_0 c^2 = T + m_0 c^2$ (see page 18 of FOBP):

$$U^{2} = p^{2}c^{2} + (m_{0}c^{2})^{2} \longrightarrow dU = vdp \longrightarrow \frac{dp}{dt} = qE_{0} = \frac{dU}{dz}$$

Accelerating gradient [MeV/m] = Change in momentum per unit time = Spatial energy gradient

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1.4 Motion in a uniform electric field (cont'd)

• Other relevant dynamical variables can be derived from knowledge of $\gamma(z)$:

$$p(z) = \beta \gamma m_0 c = \sqrt{(\gamma^2(z) - 1)} \times m_0 c$$

$$v(z) = \beta c = \frac{p(z)c^2}{U(z)} = c\sqrt{1 - \frac{1}{\gamma^2(z)}}$$

• We can also explore acceleration from the point of view of explicit time dependence:

$$\frac{d\gamma}{dt} = \beta (1 - \beta^2)^{-3/2} \frac{d\beta}{dt} = \beta \gamma^3 \frac{d\beta}{dt}$$
$$\frac{dp_z}{dt} = m_0 c \frac{d(\beta_z \gamma)}{dt} = m_0 c \left[\gamma + \beta_z^2 \gamma^3\right] \frac{d\beta_z}{dt} \simeq \gamma^3 m_0 c \frac{d\beta_z}{dt} = q E_0$$
$$\beta_z \simeq \beta$$

$$\frac{dv_z}{dt} = \frac{qE_0}{\gamma^3 m_0}$$
Longitudinal mass



[Note] Edge effects

For the case of entry into a uniform electric field with azimuthal symmetry: •



- Outward force (defocusing momentum kick) when exiting:
- No exact cancellation between focusing and defocusing momentum kicks:
 - Fields vary in time as the particles cross the gap. For longitudinal stability, the field is rising when _ the reference(synchronous) particle is injected. A field in the second half that is higher than the field in the first half, resulting in a net defocusing force: RF-defocusing force (important for ion linacs).
 - The particle velocity increases and radial position changes, while the particle crosses the gap: more important in electron linacs.



1.5 Motion in quadrupole fields

• Field free ($\mathbf{J} = 0$) vacuum region ($\mu = \mu_0$):

Sometimes (-) sign is omitted for simplicity

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = 0 \quad \longrightarrow \quad \mathbf{B} = -\nabla \psi, \quad \nabla^2 \psi = 0$$

• In the limit of a device long compared to its transverse dimensions:

$$\nabla^2 \psi \approx \nabla_{\perp}^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} = 0$$

• The solution of the above equation are of a form that is well behaved on axis (by separation of variables):

$$\psi = \sum_{n=1}^{\infty} a_n \rho^n \cos(n\phi) + b_n \rho^n \sin(n\phi)$$

 \rightarrow Be careful ! Index convention (n from 1 vs. n from 0) differs in US and Europe, and by authors and textbooks



1.5 Motion in quadrupole fields (cont'd)

• For
$$n = 1$$
:

 $\psi_{1} = a_{1}\rho\cos(\phi) + b_{1}\rho\sin(\phi) = a_{1}x + b_{1}y$ $\longrightarrow \quad \text{Equipotential surfaces form lines}$ $\mathbf{B}_{1} = -\nabla\psi_{1} = -\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)\psi_{1} = -a_{1}\hat{x} - b_{1}\hat{y}$ $\bigvee_{\text{Skew dipole}} \quad \text{Dipole}$ For n = 2: $\psi_{2} = a_{2}\rho^{2}\cos(2\phi) + b_{2}\rho^{2}\sin(2\phi) = a_{2}(x^{2} - y^{2}) + 2b_{2}xy$ $\longrightarrow \quad \text{Equipotential surfaces form hyperbolae}$ $\mathbf{B}_{2} = -\nabla\psi_{2} = -\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)\psi_{2} = 2a_{2}(-x\hat{x} + y\hat{y}) - 2b_{2}(y\hat{x} + x\hat{y})$ $\bigvee_{\text{Skew quadrupole}} \quad \text{Quadrupole}$

Dipole and Skew dipole













1.5 Motion in quadrupole fields (cont'd)

• Force due to quadrupole fields:

Please check whether the sign is correct in Eq. (2.44) of FOBP $\mathbf{F}_{\perp} = q v_z \hat{z} \times \mathbf{B}_2 = -2q v_z b_2 (y \hat{y} - x \hat{x})$

• Meaning of the coefficient b_2 : Measure of field gradient

$$-2b_2 = \frac{\partial B_x}{\partial y}\Big|_{(0,0)} = \frac{\partial B_y}{\partial x}\Big|_{(0,0)} \equiv B'$$

 Transverse equations of motion for a momentum p₀, assuming paraxial motion near the zaxis:

$$x'' = \frac{F_x}{\gamma m_0 v_0^2} = \frac{+2qv_z b_2 x}{\gamma m_0 v_0^2} = -\frac{qB'}{p_0} x$$
$$y'' = \frac{F_y}{\gamma m_0 v_0^2} = \frac{-2qv_z b_2 y}{\gamma m_0 v_0^2} = +\frac{qB'}{p_0} y$$

• In standard oscillator form:

$$x'' + \kappa_0^2 x = 0, \quad y'' - \kappa_0^2 y = 0$$

• Here, the square wave number is sometimes known as the focusing strength:

$$\kappa_0^2 \equiv \frac{qB'}{p_0} = K$$



1.5 Motion in quadrupole fields (cont'd)

For κ₀² > 0, one has simple harmonic oscillation in *x* (around *x=0*), and the motion in *y* is hyperbolic.

$$x = x_0 \cos \left[\kappa_0(z - z_0)\right] + \frac{x'_0}{\kappa_0} \sin \left[\kappa_0(z - z_0)\right] \quad \text{with } x(z_0) = x_0, \quad x'(z_0) = x'_0$$
$$y = y_0 \cosh \left[\kappa_0(z - z_0)\right] + \frac{y'_0}{\kappa_0} \sinh \left[\kappa_0(z - z_0)\right] \quad \text{with } y(z_0) = y_0, \quad y'(z_0) = y'_0$$

- For $\kappa_0^2 < 0$, the motion is simple harmonic(oscillatory) in *y*, and hyperbolic(unbounded) in *x*.
- Focusing with quadrupoles alone can only be accomplished in one transverse direction at a time. Ways of circumventing this apparent limitation in achieving transverse stability, by use of alternating gradient focusing.



[Example]



• Field varies linearly

+ Current In



 $B_x = B'y, \quad B_y = B'x \quad (B' < 0 \text{ for this Fig.})$

$$\mathbf{B} = B' \left(y \hat{x} + x \hat{y} \right)$$
$$\mathbf{A} = -\frac{1}{2} B' \left(x^2 - y^2 \right) \hat{z}$$

A standard technique for insulating magnet coils is to use epoxy resin, reinforced with fiberglass.

The in and out conductors should be placed close to each other so that longitudinal fields are minimized.



[Note] Electric quadrupole

- The commonly encountered level of 1 T static magnetic field is equivalent to a 299.8 MV/m static electric field in force for a relativistic ($v \approx c$) charged particle.
- This electric field exceeds typical breakdown limits on metallic surfaces by nearly two orders of magnitude, giving partial explanation to the predominance of magnetostatic devices over electrostatic devices for manipulation of charged particle beams.
- Therefore, the transverse electric field quadrupole is found mainly in very low energy applications.



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