Eigen-emittance and Beam matching

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- Eigen-emittance
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A few words on beam matching

[Stanley Humphries, Jr.]

A beam matched to a periodic focusing system has envelope oscillations of minimum amplitude. Furthermore, we shall find in Section 4.4 that the emittance growth caused by lens non-linearities is smallest for a matched beam. We outlined a numerical method to find matched beam distributions in Section 3.7. In this section, we shall study analytic methods that use

In a particle simulation involving a periodic lattice, it is usually desired to generate particles in a *matched* state, which means that the shape of the distribution should not change after one passage through the lattice. In fact, if a matched distribution can be found, one often has already accomplished a great deal in the understanding of the simulation. Additionally, there are circumstances in which the knowledge of the effective emittance and optics parameters

[Malte Titze]

The matched beam envelope is the solution to the KV envelope equations with the periodicity of the focusing lattice. The matched solution is generally believed to have the smallest maximum radial excursions relative to other possible envelope evolutions in the lattice and it requires particular initial conditions in the envelope of beam

[Steve Lund]

Beam matching in 2+ D

Continuing the discussion of periodic beam lines, the next step is to introduce the concept of a *matched distribution*. A matched distribution at any point in a periodic beam line is a phase space distribution of particles that is unchanged after the bunch is transported along one periodic section of the beam line. For the present purposes, we need consider only the second-order moments of the distribution: we do not need to specify whether the distribution is uniform, parabolic, Gaussian or some more exotic function. For convenience, we define the $2n \times 2n$ matrix Σ (in n degrees of freedom) with elements Σ_{ij} defined by:

$$\Sigma_{ij} = \langle x_i x_j \rangle, \tag{5.105}$$

$$R\Sigma R^{\mathrm{T}} = \sum_k N^{-1} T^k (N^{-1})^{\mathrm{T}} \epsilon_k = \Sigma. \tag{5.122}$$

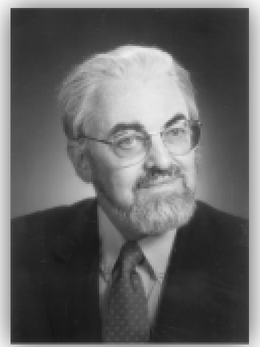
Therefore, a matrix Σ constructed using (5.116) is unchanged under transport through one periodic section of the beam line. In other words, such a matrix represents a matched distribution. Note that this is true for any values of the emittances ϵ_k: there are infinitely many matched distributions for a given beam line, although the number of degrees of freedom in choosing a matched distribution is only equal to the number of degrees of freedom in the particle motion.

[Andrzej Wolski]

From the textbook

→ The so-called Courant-Snyder invariant:

$$
\begin{aligned}
I_{CS} &= \gamma(s)x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \\
&= \begin{pmatrix} x & x' \end{pmatrix} \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \\
&= \begin{pmatrix} x \\ x' \end{pmatrix}^T \begin{pmatrix} 1/w & 0 \\ -w' & w \end{pmatrix}^T \begin{pmatrix} 1/w & 0 \\ -w' & w \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \\
&= \bar{z}^T \bar{z} = \text{const.}
\end{aligned}
$$



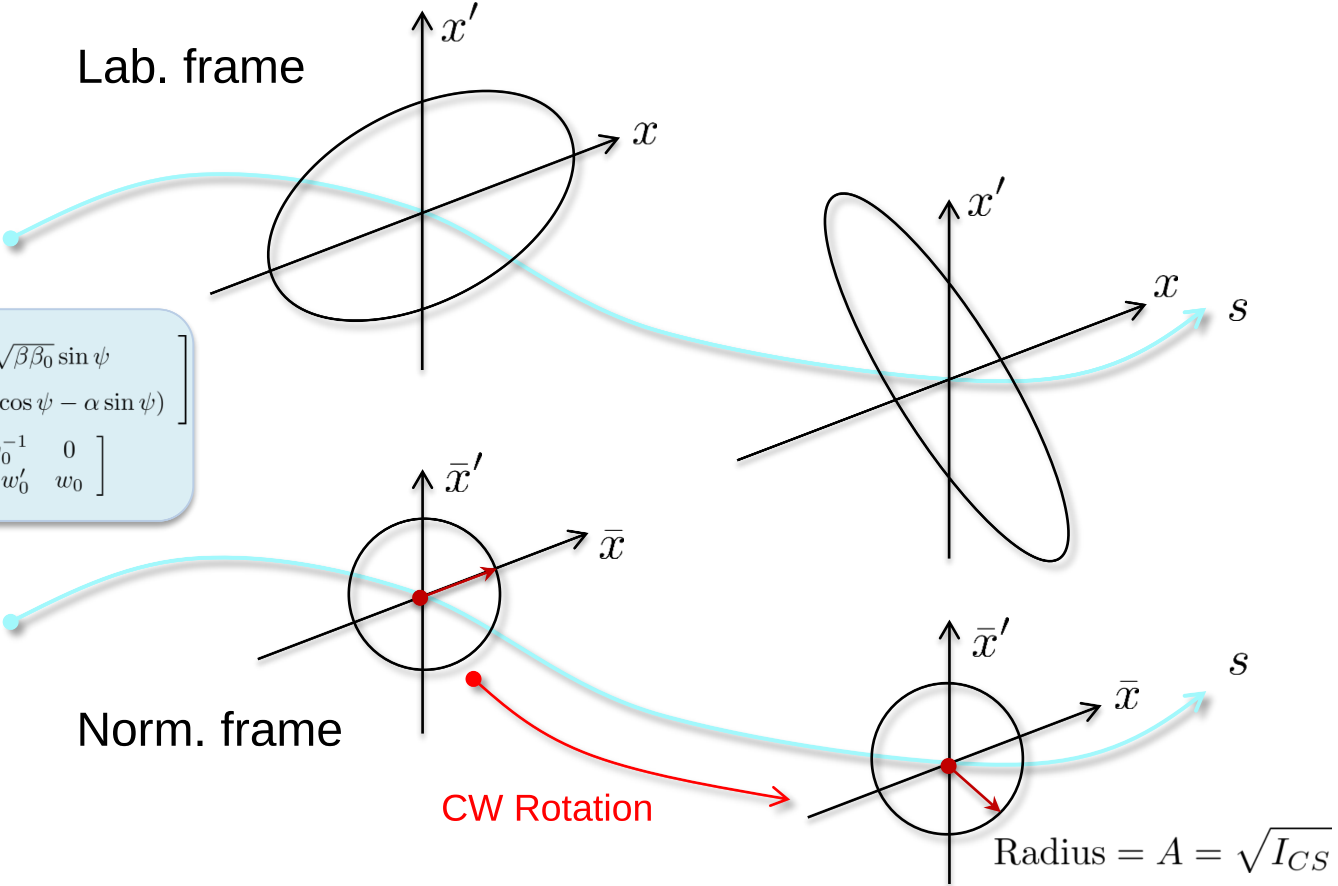
Ernest Courant (1958)

$$
\begin{aligned}
\bar{z} &= \begin{pmatrix} \bar{x} \\ \bar{x}' \end{pmatrix} = \text{normalized coordinates} \\
&= \begin{pmatrix} 1/w & 0 \\ -w' & w \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \\
&= Q(s)z(s) \\
&\neq \text{const.}
\end{aligned}
$$

Normalized Coordinates

$$
\begin{aligned}
M(s) &= \begin{bmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\psi + \alpha_0 \sin\psi) & \sqrt{\beta\beta_0}\sin\psi \\ -\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\psi + \frac{\alpha_0-\alpha}{\sqrt{\beta\beta_0}}\cos\psi & \sqrt{\frac{\beta_0}{\beta}}(\cos\psi - \alpha\sin\psi) \end{bmatrix} \\
&= \begin{bmatrix} w & 0 \\ w' & w^{-1} \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} w_0^{-1} & 0 \\ -w_0' & w_0 \end{bmatrix}
\end{aligned}
$$

CW Rotation



Another Invariant

→ By counter-acting the rotation, we can make the coordinates unchanged:

$$\begin{pmatrix} \bar{\bar{x}} \\ \bar{\bar{x}}' \end{pmatrix} = \bar{\bar{z}} = P(s)\bar{z}(s) = P(s)Q(s)z(s) = \text{const.} = P_0 Q_0 z_0 = P_0 \bar{z}_0 = \bar{\bar{z}}_0 = \begin{pmatrix} \bar{\bar{x}}_0 \\ \bar{\bar{x}}'_0 \end{pmatrix}$$

Here,

$$P(s) = \begin{pmatrix} \cos\psi(s) & -\sin\psi(s) \\ \sin\psi(s) & \cos\psi(s) \end{pmatrix}, \quad \psi' = \frac{1}{w^2}$$

Phase advance matrix (CCW rotation) Phase advance rate

$$P(0) = P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

No need to counter-act the rotation at s = 0

[Simple Proof]

→ Proof of the statement in the previous page:

Amplitude ↗ A, Initial phase ↗ ϕ_0

$$x(s) = Aw(s)\cos[\psi(s) + \phi_0]$$

→ By directly insert this in the coordinate transformation,

$$\begin{aligned}
\begin{pmatrix} \bar{\bar{x}} \\ \bar{\bar{x}}' \end{pmatrix} &= \begin{bmatrix} \cos\psi \times w^{-1}x - \sin\psi \times (-w'x + wx) \\ \sin\psi \times w^{-1}x + \cos\psi \times (-w'x + wx) \end{bmatrix} \\
&= \begin{pmatrix} A\cos\phi_0 \\ -A\sin\phi_0 \end{pmatrix} \\
&= \text{const.}
\end{aligned}$$

Hence,

$$\bar{\bar{z}} = P(s)Q(s)z(s) = \text{const.}$$

New Form of Invariant

Therefore,

$$I_{CS} = \bar{\bar{z}}^T \bar{\bar{z}} = \text{const.}$$

In fact, there can exist other "Quadratic" Invariants, such as

$$I_\xi = \bar{\bar{z}}^T \xi \bar{\bar{z}} = \text{const.} \geq 0$$

$$\xi \quad = \quad \text{a } 2 \times 2 \text{ constant positive definite (symmetric) matrix}$$

Beam Distribution Function

1) Any positive-definite (because it should represent particle counts) distribution function formed from a set of single-particle constants of the motion (C_i) will produce a valid, exact equilibrium solution to the Vlasov equation [Seteve lund]:

$$\frac{d}{ds}f(\{C_i\}) = 0$$

2) However, the Gaussian distribution is commonly used [S. Y. Lee]:

$$f(z) = \frac{1}{(2\pi)^{D/2}}\frac{1}{|\Sigma|}\times\exp\left\{-\frac{1}{2}z^T\Sigma^{-1}z\right\},\quad \Sigma = \langle zz^T\rangle = \text{beam (covariant) matrix}$$

D: dimension of z

From 1) & 2)

$$\longrightarrow \frac{1}{(2\pi)^{D/2}}\frac{1}{|\Sigma|}\times\exp\left\{-\frac{1}{2}I_\xi\right\},\quad I_\xi = \bar{\bar{z}}^T\xi\bar{\bar{z}} = z^TQ^TP^T\xi PQz$$

$$\longrightarrow \Sigma^{-1} = Q^TP^T\xi PQ,\ \text{or}\ \ \Sigma = Q^{-1}P^{-1}\varepsilon P^{-T}Q^{-T},\ \text{with}\ \ \varepsilon = \xi^{-1}$$

Sort of Matching #1

Beam Distribution in Normalized Coordinates

At s = 0

$$\begin{aligned}\bar{\Sigma}(0) &= \left\langle \bar{z}\bar{z}^T \right\rangle_0 \\ &= Q_0 \left\langle zz^T \right\rangle_0 Q_0^T \\ &= Q_0 \Sigma(0) Q_0^T \\ &\rightarrow Q_0 \left[Q_0^{-1} P_0^{-1} \varepsilon P_0^{-T} Q_0^{-T} \right] Q_0^T \\ &= \varepsilon\end{aligned}$$

Without loss of generality,
(phase advance is measured from s=0)

$$P_0 = I$$

At s > 0

$$\begin{aligned}\bar{\Sigma}(s) &= \left\langle \bar{z}\bar{z}^T \right\rangle \\ &= P^{-1} \bar{\Sigma}(0) P^{-T} \\ &= P^{-1} \varepsilon P^{-T}\end{aligned}$$

$$\varepsilon = \epsilon \begin{pmatrix} \bar{\beta} & 0 \\ 0 & 1/\bar{\beta} \end{pmatrix}$$

Without loss of generality, ε can be written in terms of diagonal matrix:
→ we choose s=0 when initial ellipse is upright, and then apply rotation
→ two parameters are required to define upright ellipse

Evolution of Beam Distribution

Without any filamentation:

$$\begin{aligned}\det\left[\bar{\Sigma}(s)\right] &= \det\left[P^{-1}\varepsilon P^{-T}\right] \\ &= \det\left[\varepsilon\right] \\ &= \epsilon^2\end{aligned}$$

With filamentation:

1) If $\bar{\beta} = 1$, beam distribution is not affected by the phase advance

$$\bar{\Sigma}(s) = \epsilon I \longrightarrow \det\left[\bar{\Sigma}(s)\right] = \epsilon^2$$

2) If $\bar{\beta} \neq 1$

$$\bar{\Sigma}(s) = \epsilon \begin{bmatrix} \bar{\beta}\cos^2\psi + \frac{1}{\bar{\beta}}\sin^2\psi & -\frac{\cos\psi\sin\psi}{\bar{\beta}} + \bar{\beta}\cos\psi\sin\psi \\ -\frac{\cos\psi\sin\psi}{\bar{\beta}} + \bar{\beta}\cos\psi\sin\psi & \bar{\beta}\sin^2\psi + \frac{1}{\bar{\beta}}\cos^2\psi \end{bmatrix}$$

After Filamentation

$$\bar{\Sigma}(s) = \epsilon \begin{bmatrix} \bar{\beta}\cos^2\psi + \frac{1}{\bar{\beta}}\sin^2\psi & -\frac{\cos\psi\sin\psi}{\bar{\beta}} + \bar{\beta}\cos\psi\sin\psi \\ -\frac{\cos\psi\sin\psi}{\bar{\beta}} + \bar{\beta}\cos\psi\sin\psi & \bar{\beta}\sin^2\psi + \frac{1}{\bar{\beta}}\cos^2\psi \end{bmatrix}$$

$$\rightarrow \epsilon \begin{bmatrix} \bar{\beta}\frac{1}{2} + \frac{1}{\bar{\beta}}\frac{1}{2} & 0 \\ 0 & \bar{\beta}\frac{1}{2} + \frac{1}{\bar{\beta}}\frac{1}{2} \end{bmatrix}$$

Average over randomly-distributed ψ

$$\sqrt{\det\left[\bar{\Sigma}(s)\right]} = \epsilon\frac{1}{2}\left(\bar{\beta} + \frac{1}{\bar{\beta}}\right) \geq \epsilon\sqrt{\bar{\beta}\frac{1}{\bar{\beta}}} = \epsilon$$

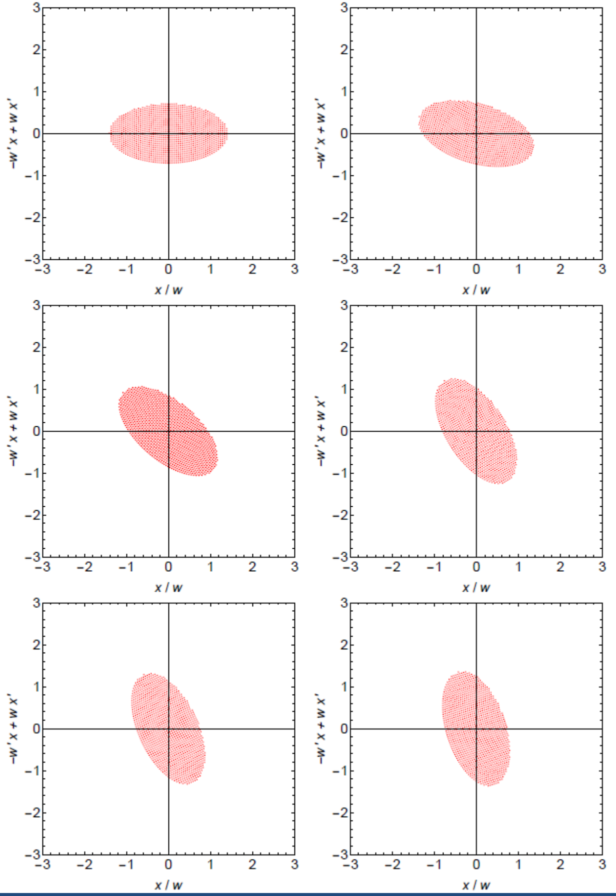
Equality for $\bar{\beta} = 1$

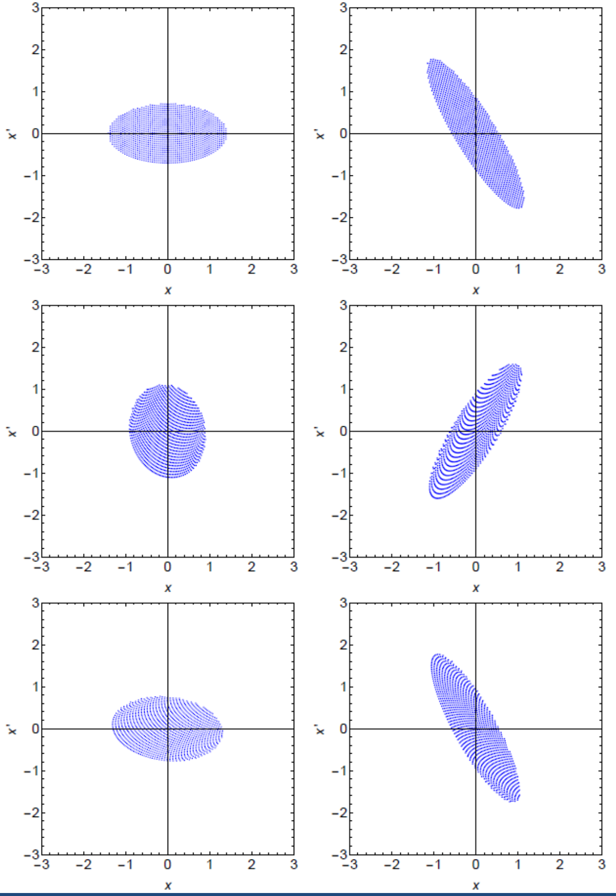
Sort of Matching #2

$$\Sigma = Q^{-1}P^{-1}\varepsilon P^{-T}Q^{-T} \longrightarrow Q^{-1}\epsilon I Q^{-T}$$

[Example: $\bar{\beta} \neq 1$]

Normalized coordinates

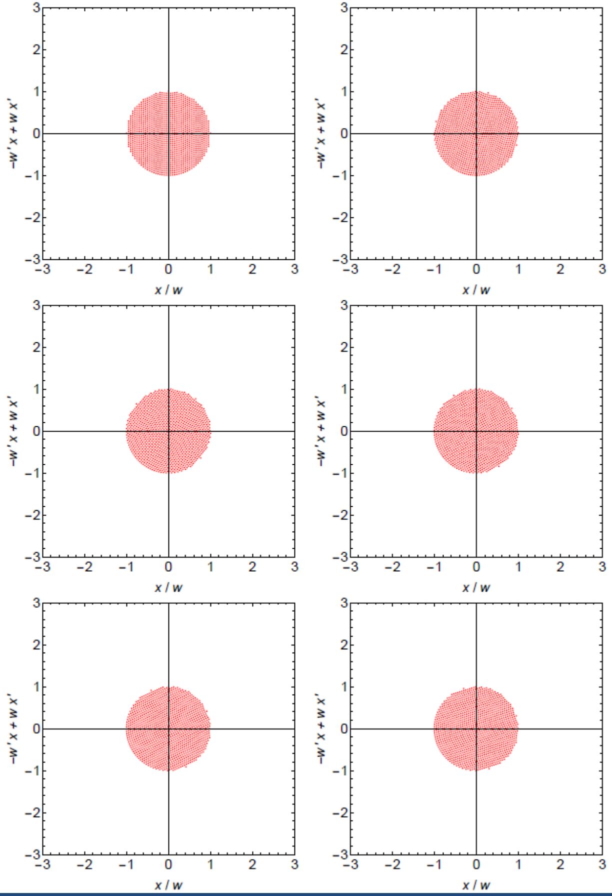


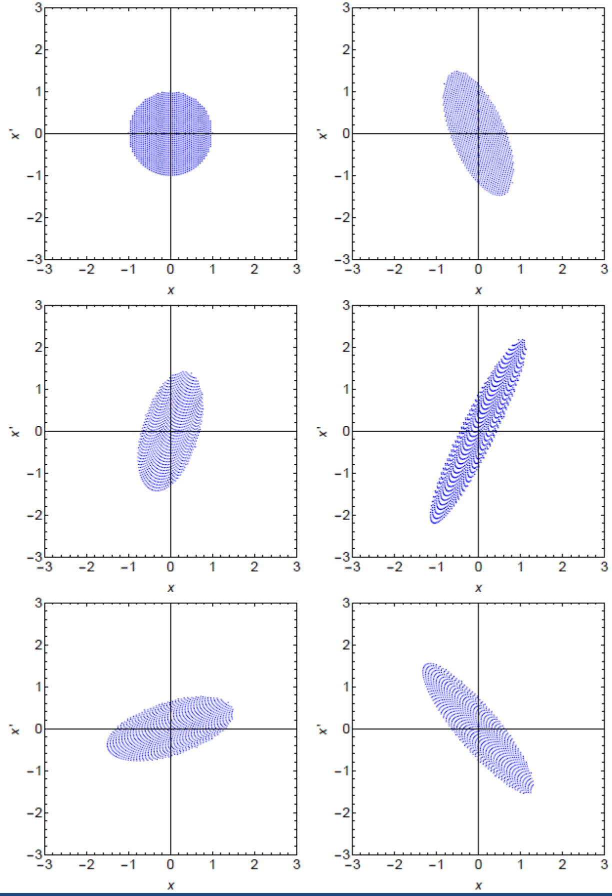


Laboratory coordinates

[Example: $\bar{\beta} = 1$]

Normalized coordinates

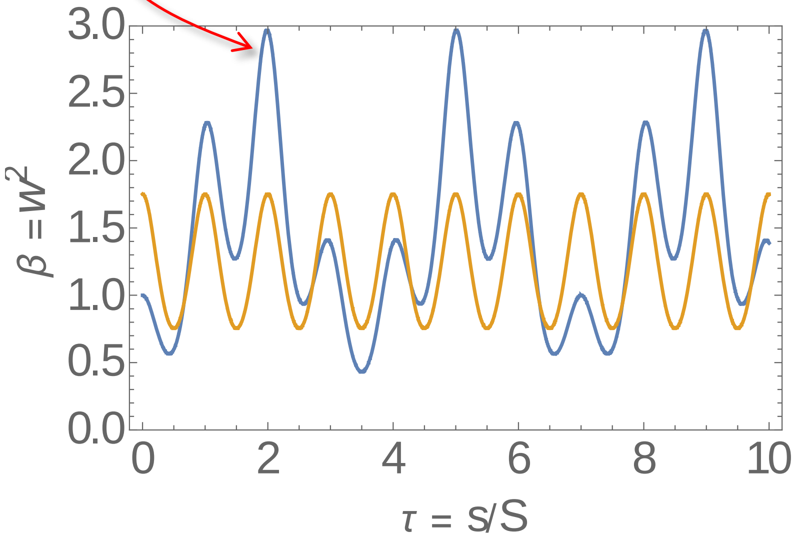




Laboratory coordinates

Periodic Matching VS Mismatching

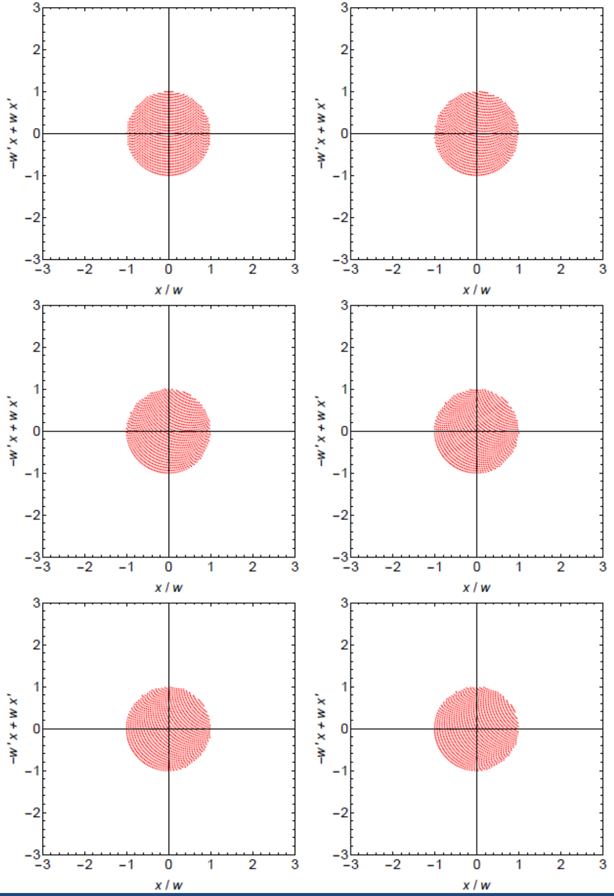
→ Plots in the previous page were made with periodically mis-matched launching condition

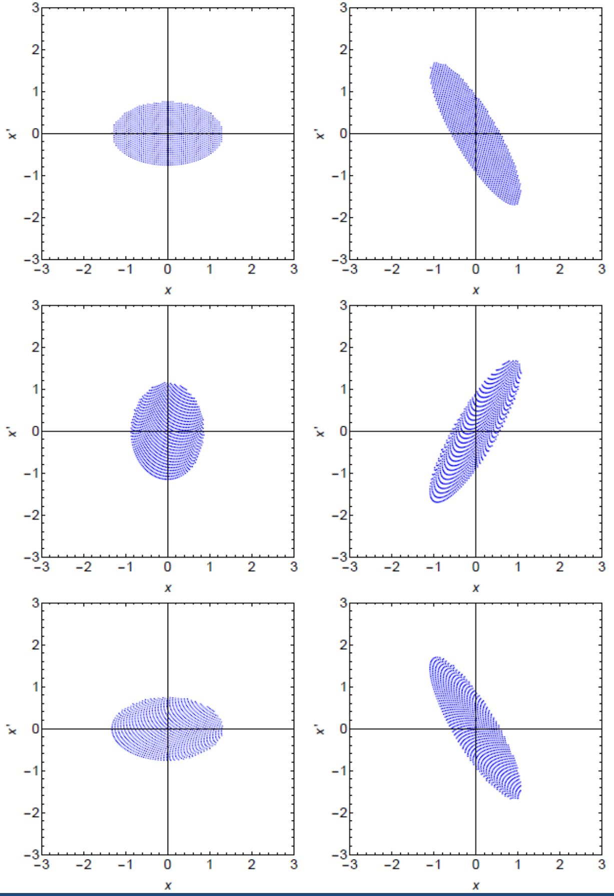


Periodically matched solution has minimum radial excursion

[Example: $\bar{\beta} = 1$ and periodic matching]

Normalized coordinates





Laboratory coordinates

Sort of Matching #3

Periodic Beam Distribution

In the periodic focusing system, the particle distribution is non-stationary, however, when plotted in trace space once per period (i.e., in the Poincare plot), we can treat the beam in stationary equilibrium.

$$
\begin{aligned}
f(s) = f(s+L) \quad &\rightarrow \quad \Sigma(s) = \Sigma(s+L) \\
&\rightarrow \quad Q(s) = Q(s+L) \\
&\rightarrow \quad w(s) = w(s+L)
\end{aligned}
$$

Already Discussed Since ~2000

Beam parameterization and invariants in a periodic solenoidal channel*

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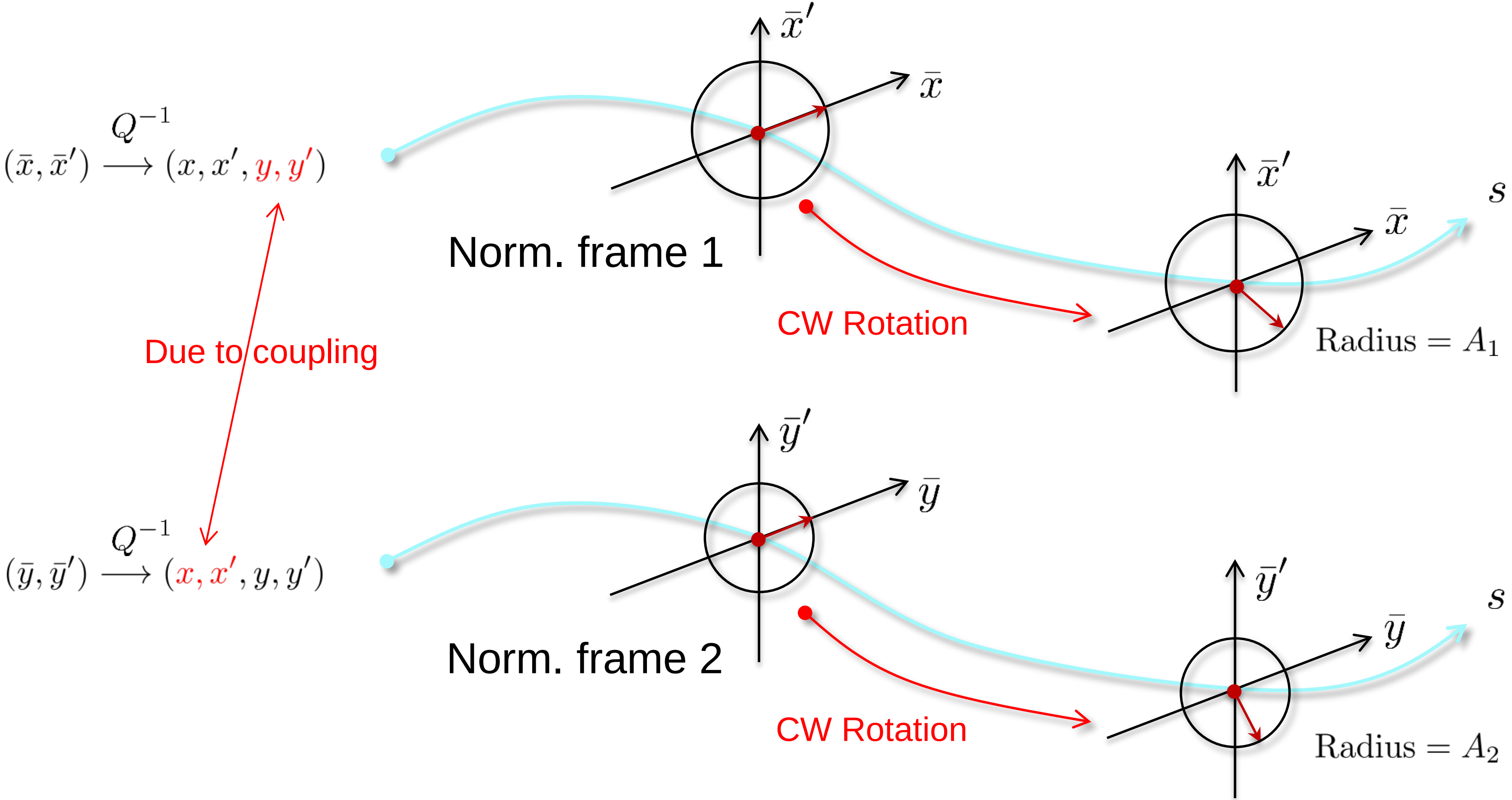
For 2+ Dimension Case



Williamson's Theorem

Diagonalization of an every 2n x 2n real, symmetric, positive definite matrix

Different from usual Eigen-decomposition

$$\Sigma = SDS^T = S \begin{bmatrix} \epsilon_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix} S^T$$

$$= S \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} S^T$$

$$S^T J S = J,\ S J S^T = J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}$$

A symplectic matrix unique up to
a unitary matrix (a symplectic rotation)
→But, not every unitary matrix can be used here
→The unitary matrix should have a special form. See slide 29

$$U(n) = \mathrm{Sp}(2n, R) \cap O(2n)$$

$$S \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} S^T = S' \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} S'^T = SU \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} U^T S^T$$

Diagonal elements are "symplectic eigenvalues" (symplectic spectrum) of Σ

$$\det [J\Sigma \pm i\lambda I] = 0$$

Eigen-emittance

$$\det[\Sigma] = \det[SDS^T] = \det[S]\det[D]\det[S^T] = \det[D] = (\epsilon_1\epsilon_2)^2$$

$$\begin{aligned} \mathrm{tr}[(\Sigma J)^2] &= \mathrm{tr}[SDS^TJ\cdot SDS^TJ] \\ &= \mathrm{tr}[SD\cdot S^TJS\cdot DS^TJ] \\ &= \mathrm{tr}[SD\cdot J\cdot DS^TJ] \\ &= \mathrm{tr}[DJ\cdot DS^TJS] \\ &= \mathrm{tr}[DJ\cdot DJ] \\ &= \mathrm{tr}[(DJ)^2] \\ &= -2(\epsilon_1^2+\epsilon_2^2) \end{aligned}$$

$$\mathrm{tr}[ABC] = \mathrm{tr}[BCA] = \mathrm{tr}[CAB]$$

$$\epsilon_{1,2} = \frac{1}{2}\sqrt{-\mathrm{tr}[(DJ)^2] \pm \sqrt{\mathrm{tr}^2[(DJ)^2] - 16\det[D]}}$$

Invariant under symplectic transformation

RMS- vs Eigen-emittances

From Fischer's inequality:

$$\Sigma = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix} \quad \longrightarrow \quad \det[\Sigma] \leq \det[A]\det[B]$$

$$\longrightarrow \quad (\epsilon_1\epsilon_2)^2 \leq \epsilon_{\mathrm{rms},x}^2 \times \epsilon_{\mathrm{rms},y}^2$$

Equality if and only if
C = 0

From direct calculation (e.g., with the help of Mathematica):

$$\begin{aligned} -\frac{1}{2}\mathrm{tr}[(\Sigma J)^2] &= \det[A] + \det[B] + 2\det[C] \\ &= \epsilon_{\mathrm{rms},x}^2 + \epsilon_{\mathrm{rms},y}^2 + 2\begin{vmatrix} \langle xy \rangle & \langle xy' \rangle \\ \langle x'y \rangle & \langle x'y' \rangle \end{vmatrix} \\ &= \epsilon_1^2 + \epsilon_2^2 \end{aligned}$$

* In this slide, we use the notation of $z = (x, x', y, y')^T$

[Example]

For a round beam in solenoids with finite average canonical angular momentum:
(i.e., distribution function is independent of angle is X-Y plane)

Using canonical coordinates

$$\Sigma = \begin{bmatrix} \langle X^2 \rangle & \langle XX' \rangle & 0 & \langle XY' \rangle \\ \langle XX' \rangle & \langle X'^2 \rangle & -\langle XY' \rangle & 0 \\ 0 & -\langle XY' \rangle & \langle X^2 \rangle & \langle XX' \rangle \\ \langle XY' \rangle & 0 & \langle XX' \rangle & \langle X'^2 \rangle \end{bmatrix}$$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\langle XY' - YX' \rangle \\ &= \langle XY' \rangle \\ &= -\langle YX' \rangle \end{aligned}$$

$$\begin{aligned} \sqrt{\det[\Sigma]} &= \langle X^2 \rangle \langle X'^2 \rangle - \langle XX' \rangle^2 - \langle XY' \rangle^2 = \epsilon_{\rm rms}^2 - \mathcal{L}^2 \\ &= \epsilon_1 \epsilon_2 \end{aligned}$$

$$\begin{aligned} -\frac{1}{2}{\rm tr}[(\Sigma J)^2] &= 2\left(\langle X^2 \rangle \langle X'^2 \rangle - \langle XX' \rangle^2 + \langle XY' \rangle^2\right) = 2\left(\epsilon_{\rm rms}^2 + \mathcal{L}^2\right) \\ &= \epsilon_1^2 + \epsilon_2^2 \end{aligned}$$

$$\epsilon_{1,2} = \epsilon_{\rm rms} \pm \mathcal{L}$$

* In this slide, we use the notation of $Z = (X, X', Y, Y')^T$ which is canonical in the Larmor frame

Already Discussed Since ~2000

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS, VOLUME **6**, 104002 (2003)

Round-to-flat transformation of angular-momentum-dominated beams

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(Received 13 June 2003; published 30 October 2003)

A study of round-to-flat configurations, and vice versa, of angular-momentum-dominated beams is presented. The beam propagation in an axial magnetic field is described in terms of the familiar Courant-Snyder formalism by using a rotating coordinate system. The discussion of the beam transformation is based on the general properties of a cylindrically symmetric beam matrix and the existence of two invariants for a symplectic transformation in 4D phase space.

DOI: 10.1103/PhysRevSTAB.6.104002 PACS numbers: 29.27.–a, 41.75.Lx, 41.85.–p

Experimental Demonstration

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS **7**, 123501 (2004)

Generation of angular-momentum-dominated electron beams from a photoinjector

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(Received 2 November 2004; published 22 December 2004)

Various projects under study require an angular-momentum-dominated electron beam generated by a photoinjector. Some of the proposals directly use the angular-momentum-dominated beams (e.g., electron cooling of heavy ions), while others require the beam to be transformed into a flat beam (e.g., possible electron injectors for light sources and linear colliders). In this paper we report our experimental study of an angular-momentum-dominated beam produced in a photoinjector, addressing the dependencies of angular momentum on initial conditions. We also briefly discuss the removal of angular momentum. The results of the experiment, carried out at the Fermilab/NICADD Photoinjector Laboratory, are found to be in good agreement with theoretical and numerical models.

DOI: 10.1103/PhysRevSTAB.7.123501 PACS numbers: 29.27.–a, 41.85.–p, 41.75.Fr

PRL **113**, 264802 (2014) PHYSICAL REVIEW LETTERS week ending 31 DECEMBER 2014

Experimental Proof of Adjustable Single-Knob Ion Beam Emittance Partitioning

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The performance of accelerators profits from phase-space tailoring by coupling of degrees of freedom. Previously applied techniques swap the emittances among the three degrees but the set of available emittances is fixed. In contrast to these emittance exchange scenarios, the emittance transfer scenario presented here allows for arbitrarily changing the set of emittances as long as the product of the emittances is preserved. This Letter is the first experimental demonstration of transverse emittance transfer along an ion beam line. The amount of transfer is chosen by setting just one single magnetic field value. The envelope functions (beta) and slopes (alpha) of the finally uncorrelated and repartitioned beam at the exit of the transfer line do not depend on the amount of transfer.

DOI: 10.1103/PhysRevLett.113.264802 PACS numbers: 41.75.Ak, 41.85.Ct, 41.85.Ja, 41.85.Lc

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Applying steps in 1D matching

From sort of Matching #1

$$\Sigma = \langle zz^T \rangle \longrightarrow Q^{-1}P^{-1}\varepsilon P^{-T}Q^{-T}, \text{ with } \varepsilon = \bar{\Sigma}(0)$$

→In principle, ε can be an arbitrary positive definite matrix

A unitary matrix (symplectic rotation) in 2+ D

But to meet the matching condition 2, ε should have a special form.

From sort of Matching #2

$$P^{-1}\varepsilon P^{-T} = \varepsilon$$

→ This should be independent of particle's phase advance

Two possible cases:

$$\varepsilon = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon \end{bmatrix}, \text{ or } \varepsilon = \begin{bmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{bmatrix} \text{ with } \theta = 0$$

θ : defined in the next slide

No motion accros the eigen-planes

From sort of Matching #3

$$Q(s) = Q(s+L)$$

[Some proof]

→ An arbitrary unitary matrix can be parametrized as (e.g., based on Sec. 3.3 of Sakurai)

$$U(1) = e^{i\theta} \leftrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Overall phase

Euler rotations

Pauli matrices

$$\begin{aligned} U(2) &= e^{i\lambda} R(\alpha,\beta,\gamma) \\ &= e^{i\lambda} \exp(-i\sigma_3\alpha/2)\exp(-i\sigma_2\beta/2)\exp(-i\sigma_3\gamma/2) \\ &= e^{i\lambda} \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \end{aligned}$$

$$\mapsto \begin{pmatrix} \cos[\lambda] & 0 & -\sin[\lambda] & 0 \\ 0 & \cos[\lambda] & 0 & -\sin[\lambda] \\ \sin[\lambda] & 0 & \cos[\lambda] & 0 \\ 0 & \sin[\lambda] & 0 & \cos[\lambda] \end{pmatrix} \begin{pmatrix} \cos[(\xi+\eta)/2] & 0 & -\sin[(\xi+\eta)/2] & 0 \\ 0 & \cos[(\xi+\eta)/2] & 0 & \sin[(\xi+\eta)/2] \\ \sin[(\xi+\eta)/2] & 0 & \cos[(\xi+\eta)/2] & 0 \\ 0 & -\sin[(\xi+\eta)/2] & 0 & \cos[(\xi+\eta)/2] \end{pmatrix}$$

$$\times \begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 & 0 \\ \sin[\theta] & \cos[\theta] & 0 & 0 \\ 0 & 0 & \cos[\theta] & -\sin[\theta] \\ 0 & 0 & \sin[\theta] & \cos[\theta] \end{pmatrix} \begin{pmatrix} \cos[(\xi-\eta)/2] & 0 & -\sin[(\xi-\eta)/2] & 0 \\ 0 & \cos[(\xi-\eta)/2] & 0 & \sin[(\xi-\eta)/2] \\ \sin[(\xi-\eta)/2] & 0 & \cos[(\xi-\eta)/2] & 0 \\ 0 & -\sin[(\xi-\eta)/2] & 0 & \cos[(\xi-\eta)/2] \end{pmatrix} = P$$

Here, $\alpha = -(\xi+\eta), \beta/2 = \theta, \gamma = \eta - \xi$

[Some proof - Continued]

$$
P^{-1}\varepsilon P^{-T} = P^{-1}\begin{bmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{bmatrix} P^{-T}
$$

$$
= \begin{pmatrix} \mathrm{Cos}[\theta]^2\epsilon_1 + \mathrm{Sin}[\theta]^2\epsilon_2 & \frac{1}{2}\mathrm{Cos}[\zeta-\eta]\mathrm{Sin}[2\theta]\left(-\epsilon_1+\epsilon_2\right) & 0 & \frac{1}{2}\mathrm{Sin}[\zeta-\eta]\mathrm{Sin}[2\theta]\left(-\epsilon_1+\epsilon_2\right) \\ \frac{1}{2}\mathrm{Cos}[\zeta-\eta]\mathrm{Sin}[2\theta]\left(-\epsilon_1+\epsilon_2\right) & \mathrm{Sin}[\theta]^2\epsilon_1 + \mathrm{Cos}[\theta]^2\epsilon_2 & \frac{1}{2}\mathrm{Sin}[\zeta-\eta]\mathrm{Sin}[2\theta]\left(\epsilon_1-\epsilon_2\right) & 0 \\ 0 & \frac{1}{2}\mathrm{Sin}[\zeta-\eta]\mathrm{Sin}[2\theta]\left(\epsilon_1-\epsilon_2\right) & \mathrm{Cos}[\theta]^2\epsilon_1 + \mathrm{Sin}[\theta]^2\epsilon_2 & \frac{1}{2}\mathrm{Cos}[\zeta-\eta]\mathrm{Sin}[2\theta]\left(-\epsilon_1+\epsilon_2\right) \\ \frac{1}{2}\mathrm{Sin}[\zeta-\eta]\mathrm{Sin}[2\theta]\left(-\epsilon_1+\epsilon_2\right) & 0 & \frac{1}{2}\mathrm{Cos}[\zeta-\eta]\mathrm{Sin}[2\theta]\left(-\epsilon_1+\epsilon_2\right) & \mathrm{Sin}[\theta]^2\epsilon_1 + \mathrm{Cos}[\theta]^2\epsilon_2 \end{pmatrix}
$$

Two possible cases that make the above expression independent of the phase advance:

$$
\longrightarrow \begin{bmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{bmatrix} \text{with } \theta = 0
$$

$$
\longrightarrow \begin{bmatrix} (\epsilon_1+\epsilon_2)/2 & 0 & 0 & 0 \\ 0 & (\epsilon_1+\epsilon_2)/2 & 0 & 0 \\ 0 & 0 & (\epsilon_1+\epsilon_2)/2 & 0 \\ 0 & 0 & 0 & (\epsilon_1+\epsilon_2)/2 \end{bmatrix} \text{with } \theta = \text{random}
$$

For special case:

If $\theta = 0(\text{or, } \beta/2 = 0)$

$$U(2) = e^{i\lambda} \begin{pmatrix} e^{-i(\alpha+\gamma)/2} & 0 \\ 0 & e^{i(\alpha+\gamma)/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i(\lambda+\xi)} & 0 \\ 0 & e^{i(\lambda-\xi)} \end{pmatrix}$$

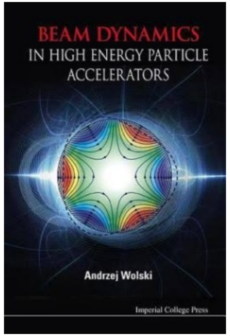
Here, $\alpha = -(\xi + \eta), \gamma = \eta - \xi, \alpha + \gamma = -2\xi$

$$\mapsto \begin{pmatrix} \cos[\lambda+\xi] & 0 & -\sin[\lambda+\xi] & 0 \\ 0 & \cos[\lambda-\xi] & 0 & -\sin[\lambda-\xi] \\ \sin[\lambda+\xi] & 0 & \cos[\lambda+\xi] & 0 \\ 0 & \sin[\lambda-\xi] & 0 & \cos[\lambda-\xi] \end{pmatrix}$$

→ This is a typical form of the double rotation.

How to calculate Q?

- No universal standard:



Coupling between horizontal and vertical motion can occur in a beam line either by design (for example, because of the inclusion of skew quadrupole or solenoid magnets), or as a result of alignment errors on the magnets (such as the tilt of a quadrupole around its magnetic axis). It is important to be able to describe coupling and its effects on the beam,

and there are several methods that have been developed to do this in a convenient way. Unfortunately, no single method has been adopted as a universal standard, and it would not be practical to try to cover here all (or even several) of the methods that are in use. Therefore, we restrict our

- Solving matrix envelope equation:

$$W = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix}, \quad W'' + \kappa W = (W^T W W^T)^{-1}, \quad Q = \begin{bmatrix} W^{-1} & 0 \\ -(W^T)' & W^T \end{bmatrix}$$

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Generalized Kapchinskij-Vladimirskij Distribution and Beam Matrix for Phase-Space Manipulations of High-Intensity Beams

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Equivalence between various methods → On-going research

Methods for Linear Coupled Optics

1) **By decoupling transformation:** directly decouple the one-turn transfer map into an uncoupled one-turn map (i.e., into a block-diagonal form) through a matrix similarity transformation

[1] D. Edwards and L. Teng, IEEE Trans. Nucl. Sci. **20**, 3 (1973).
[2] D. Sagan and D. Rubin, Phys. Rev. Accel. Beams **2**, 074001 (1999).
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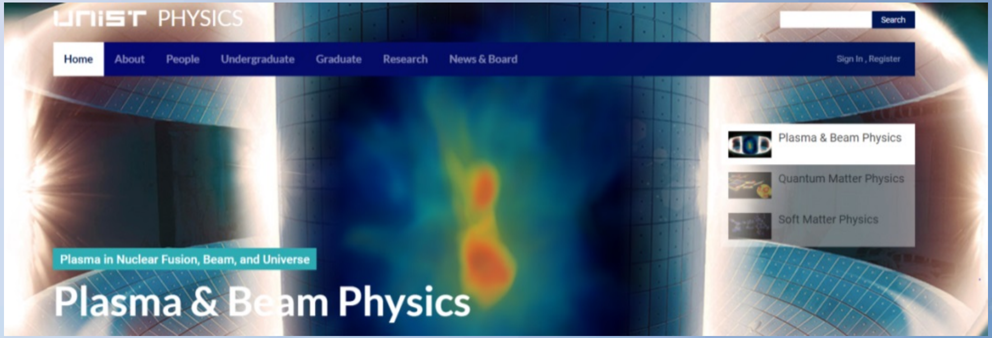
2) **Using eigenvectors of the transfer matrix:** a transformation is found from the eigenvectors of the transfer matrix that puts the transfer matrix into "normal form", i.e., the transfer matrix is transformed into a pure rotation

[1] G. Ripken, DESY Internal Report No. R1-70/04, 1970.
[2] A. V. Lebedev and S. A. Bogacz, J. Instrum. **5**, P10010 (2010).
[3] A. Wolski, Phys. Rev. Accel. Beams **9**, 024001 (2006).

Conclusions

- Beam matching in 1 D → well-established, well-known
- Eigen-emittance → well-established, not well-known
- Beam matching in 2+ D → not completely established, not well-known



Thank you for your attention !