5.8 Diffusion in Fully Ionized Plasmas

In the absence of gravity,

$$\begin{aligned}\mathbf{J}\times\mathbf{B} &= \nabla p \\ \mathbf{E}+\mathbf{v}\times\mathbf{B} &= \eta\mathbf{J}\end{aligned} \tag{5.124}$$

The parallel component of the generalized Ohm's law:

$$E_{\parallel} = \eta_{\parallel} J_{\parallel} : \quad \text{ordinary Ohm's law}$$

The perpendicular component:

$$\mathbf{E}\times\mathbf{B} + (\mathbf{v}_{\perp}\times\mathbf{B})\times\mathbf{B} = \eta_{\perp}\mathbf{J}\times\mathbf{B} = \eta_{\perp}\nabla p$$

$$\mathbf{E}\times\mathbf{B} - \mathbf{v}_{\perp}B^2 = \eta_{\perp}\nabla p$$

$$\mathbf{v}_{\perp} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} - \frac{\eta_{\perp}}{B^2}\nabla p$$

The flux associated with diffusion is

$$\mathbf{\Gamma} = n\mathbf{v}_{\perp} = -\frac{\eta_{\perp} n(KT_i + KT_e)}{B^2}\nabla n \tag{5.125}$$

This has the form of Fick's law with the diffusion coefficient

$$\boxed{D_{\perp} = \frac{\eta_{\perp} n \sum KT}{B^2}} \tag{5.126}$$

This is called the classical diffusion coefficient for a fully ionized gas.

NOTES:

Fully ionized plasma	Weakly ionized plasma
$D_{\perp} = \frac{\eta_{\perp} n \sum KT}{B^2}$	$D_{\perp} = \frac{KT}{m\nu}\frac{1}{1+\omega_c^2\tau^2}$
$D_{\perp} \propto \frac{1}{B^2}$	$D_{\perp} \propto \frac{1}{B^2}$
$D_{\perp} \propto n$	$D_{\perp}$: independent of n
$D_{\perp} \propto T^{-\frac{1}{2}}$	$D_{\perp}\uparrow$ as $T\uparrow$

5.9 Solutions of the Diffusion Equation

Let

$$A \equiv \frac{\eta KT}{B^2} \tag{5.127}$$

For $T_i = T_e$,

$$D_\perp = 2nA$$

The equation of continuity leads to

$$\frac{\partial n}{\partial t} + \nabla \cdot (-D_\perp \nabla n) = 0$$

$$\frac{\partial n}{\partial t} = \nabla \cdot (D_\perp \nabla n) = A\nabla \cdot (2n\nabla n) = A\nabla^2 n^2 \tag{5.128}$$

5.9.1 Time Dependence

$$n(\mathbf{r}, t) = T(t)S(\mathbf{r})$$

$$\frac{1}{T^2}\frac{dT}{dt} = \frac{A}{S}\nabla^2 S^2 = -\frac{1}{\tau} \tag{5.129}$$

$$\frac{1}{T} = \frac{1}{T_0} + \frac{t}{\tau} \tag{5.130}$$

At large times t, $T \propto \frac{1}{t}$, as in the case of recombination. For a weakly ionized case, on the other hand, it is an exponential decay.

5.10 Time-Independent Solutions

- Consider a steady state when particles diffusing from the denser region of space are lost by recombination: In the region outside the source,

$$-A\nabla^2 n^2 = -\alpha n^2 \tag{5.131}$$

  In a slab,

$$\frac{\partial^2 n^2}{\partial x^2} = \frac{\alpha}{A} n^2 \tag{5.132}$$

$$n^2 = n_0^2 \exp[-\sqrt{\alpha/A}x] \tag{5.133}$$

- In a fully ionized steady state plasma maintained by a constant source Q in a uniform B field,

$$-A\nabla^2 n^2 = -\eta KT\nabla^2(n^2/B^2) = Q \tag{5.134}$$

  Since n and B occur only in the combination nB, we have

$$n \propto B \tag{5.135}$$

5.11 Bohm Diffusion and Neoclassical Diffusion

$$D_\perp \propto \begin{cases} \dfrac{1}{B^2} & \text{from classical theory} \\ \dfrac{1}{B} & \text{from experiments} \end{cases}$$

The semiempircial formula by Bohm:

$$D_\perp = \frac{1}{16}\frac{KT_e}{eB} \equiv D_B \gg \underbrace{D_\perp}_{classical} \tag{5.136}$$

$$\Gamma_\perp = nv_\perp \propto nE/B \tag{5.137}$$

$$e\phi_{max} \simeq KT_e \longrightarrow E_{max} \simeq \frac{\phi_{max}}{R} \simeq \frac{KT}{eR} \tag{5.138}$$

Thus

$$\Gamma_\perp \simeq \gamma\frac{n}{R}\frac{KT_e}{eB} \simeq -\gamma\frac{KT_e}{eB}\nabla n = -D_B\nabla n \tag{5.139}$$

This diffusion coefficient was proposed, not derived. $\gamma = 1/16$ empirically.

$$\tau \simeq -\frac{N}{\frac{dN}{dt}} = \frac{n\pi R^2 L}{2\pi RL\Gamma_\perp} = \frac{nR}{2\Gamma_\perp} = \frac{nR}{2D_B\nabla n} \simeq \frac{nR}{2D_B\frac{n}{R}} = \frac{R^2}{2D_B}$$

$$\tau_B = \frac{R^2}{2D_B} : \quad \text{Bohm diffusion time}$$

Proposed explanations:

- Asymmetry in the magnetic field.
- Asymmetric electric field.
- Oscillating electric field arsing from unstable plasma wave.

In a toroidal device,

Helical lines of force (to eliminate grad-B and curvature drift)
$\longrightarrow$ Banana orbit ($\longleftarrow$ mirror effect)
$\longrightarrow$ The random walk step length=the width of banana orbit
$\longrightarrow$ The classical diffusion coefficient increased (Neoclassical diffusion)

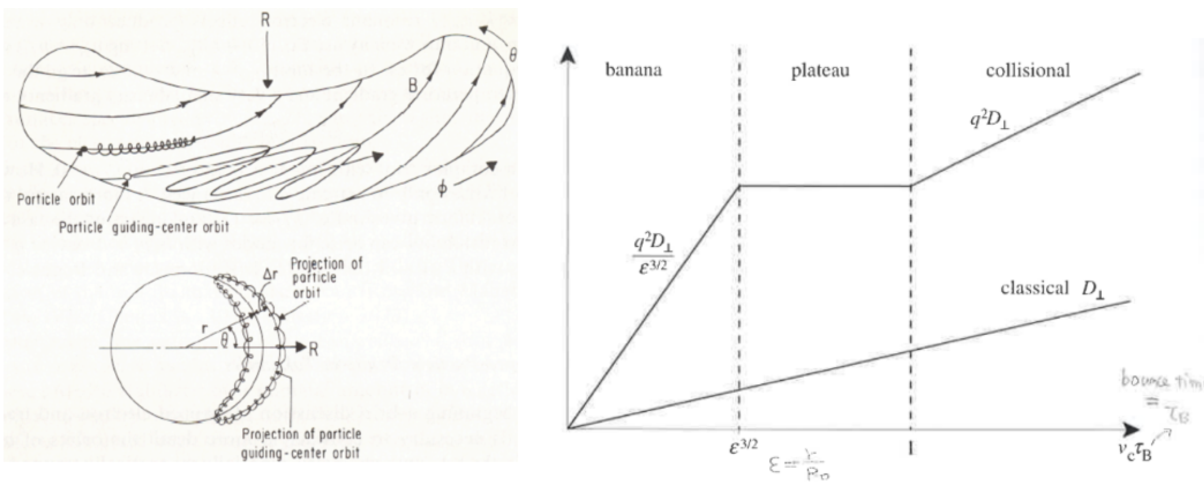

Figure 5.7: (left) Banana orbit in tokamak. (right) Variation of perpendicular diffusion coefficient with collision frequency ν_c. Here, q is the safety factor, $\varepsilon = r/R_0$ is the inverse aspect ratio, and τ_B is the bounce time for the banana orbit.