4.9 Review of Electromagnetic Waves in a Vacuum

The Maxwell equations in a vacuum are

$$\epsilon_0 \nabla \cdot \mathbf{E} = 0 \tag{4.139}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.140}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4.141}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4.142}$$

Two curl equations can be combined into one

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= -\frac{\partial}{\partial t} \nabla \times \mathbf{B} \\ &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \end{aligned}$$

Since $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ and $\mu_0 \epsilon_0 = c^{-2}$,

$$-\nabla^2 \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{4.143}$$

Assuming plane waves varying $\exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$, we have

$$k^2 \mathbf{E} = \frac{\omega^2}{c^2} \mathbf{E} \tag{4.144}$$

which leads to

$$\boxed{\omega^2 = k^2 c^2} \tag{4.145}$$

NOTES

- phase velocity: $v_p = c$
- group velocity: $v_g = c$

4.10 Electromagnetic Waves with $\mathbf{B}_0 = 0$

ASSUMPTIONS

- Homogeneous infinite quasineutral plasma
- No external field: $\mathbf{E}_0 = \mathbf{B}_0 = 0$
- Cold Plasma: $T_i = T_e = 0$

- Immobile ions: $\mathbf{v}_{i1} = 0$, $n_{i1} = 0$

FLUID EQUATIONS

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e \right] = -n_e e \mathbf{E} - e n_e \mathbf{v}_e \times \mathbf{B} \tag{4.146}$$

$$\mathbf{J} = -e n_e \mathbf{v}_e \tag{4.147}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.148}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4.149}$$

LINEARIZED EQUATIONS

$$m_e \frac{\partial \mathbf{v}_{e1}}{\partial t} = -e\mathbf{E}_1 \longrightarrow \mathbf{v}_{e1} = \frac{e\mathbf{E}_1}{i m_e \omega} \tag{4.150}$$

$$\mathbf{J}_1 = -e n_0 \mathbf{v}_{e1} = \frac{i n_0 e^2 \mathbf{E}_1}{m_e \omega} \tag{4.151}$$

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}_1) &= -\frac{\partial}{\partial t} \nabla \times \mathbf{B}_1 \\ &= -\mu_0 \frac{\partial \mathbf{J}_1}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}_1}{\partial t^2} \\ &= -\mu_0 \frac{n_0 e^2}{m_e} \mathbf{E}_1 - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}_1}{\partial t^2} \,. \end{aligned} \tag{4.152}$$

Therefore, we obtain

$$\nabla(\nabla \cdot \mathbf{E}_1) - \nabla^2 \mathbf{E}_1 = -\frac{\omega_{pe}^2}{c^2} \mathbf{E}_1 - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_1}{\partial t^2} \tag{4.153}$$

or

$$-\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1) + k^2 \mathbf{E}_1 = \frac{\omega^2 - \omega_{pe}^2}{c^2} \mathbf{E}_1 \tag{4.154}$$

$\mathbf{E}_1$ may be spilt into two parts (longitudinal part and transverse part)

$$\mathbf{E}_1 = \mathbf{E}_\parallel + \mathbf{E}_\perp \tag{4.155}$$

where $\mathbf{E}_\perp \perp \mathbf{k}$ and $\mathbf{E}_\parallel \parallel \mathbf{k}$.

DISPERSION RELATION

- Longitudinal part:

$$-k^2\mathbf{E}_{\|} + k^2\mathbf{E}_{\|} = \frac{\omega^2 - \omega_{pe}^2}{c^2}\mathbf{E}_{\| } \tag{4.156}$$

$$\boxed{\omega^2 = \omega_{pe}^2} \quad \text{plasma oscillation} \tag{4.157}$$

- Transverse part:

$$k^2\mathbf{E}_{\perp} = \frac{\omega^2 - \omega_{pe}^2}{c^2}\mathbf{E}_{\perp} \tag{4.158}$$

So the dispersion relation is given by

$$\boxed{\omega^2 = \omega_{pe}^2 + k^2c^2} \tag{4.159}$$

NOTES

For transverse waves,

- Phase velocity:

$$v_p^2 = \frac{\omega^2}{k^2} = c^2 + \frac{\omega_{pe}^2}{k^2} > c^2 \tag{4.160}$$

- Group velocity:

$$v_g = \frac{d\omega}{dk} = \frac{c^2}{v_p} < c \tag{4.161}$$

- The index of refraction: $n^2 = (\frac{c}{v_p})^2 = (\frac{kc}{\omega})^2 = 1 - \frac{\omega_{pe}^2}{\omega^2} < 1$.

- Since $k^2 = \frac{\omega^2 - \omega_{pe}^2}{c^2}$

  1. $\omega > \omega_{pe}$: k is real so that the wave is propagating.
  2. $\omega = \omega_{pe}$: $k = 0$ (cutoff)
  3. $\omega < \omega_{pe}$: k is imaginary so that the wave is evanescent.
     $e^{ikx} = e^{-|k|x} = e^{-x/\delta}$
     where $\delta = \frac{1}{|k|} = \frac{c}{\sqrt{\omega_{pe}^2 - \omega^2}}$: skin depth.

- Detecting the phase shift, the plasma density may be measured.

- The wave with $\omega < \omega_{pe}$ is reflected from the plasma.

  – It is possible to send radio waves around the earth.
  – It is necessary to use frequency above ω_{pe} to communicate with space vehicle.
  – The plasma density may be estimated from the cut-off frequency.

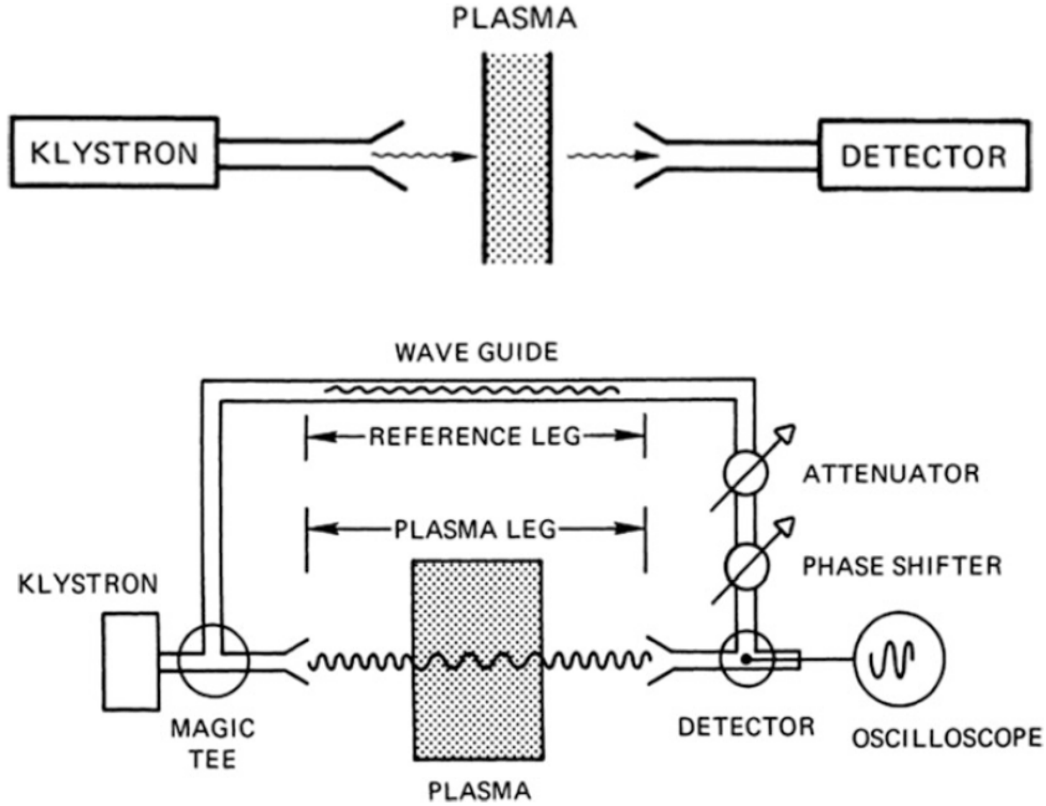

Figure 4.7: Microwave measurement of plasma density by the cutoff of the transmitted signal (top), and a microwave interferometer for plasma density measurement (bottom).

4.11 Electromagnetic Waves with $\mathbf{k} \perp \mathbf{B}_0$

4.11.1 Ordinary Waves ($\mathbf{E}_1 \parallel \mathbf{B}_0$)

ASSUMPTIONS

- Homogeneous infinite quasineutral plasma
- $\mathbf{E}_0 = 0$, $\mathbf{B}_0 \neq 0$
- Cold Plasma: $T_i = T_e = 0$
- Immobile ions: $\mathbf{v}_{i1} = 0$, $n_{i1} = 0$

FLUID EQUATIONS

$$mn_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e \right] = -ne\mathbf{E} - en_e\mathbf{v}_e \times \mathbf{B} \tag{4.162}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.163}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4.164}$$

with

$$\mathbf{J} = -en_0 \mathbf{v}_e \tag{4.165}$$

LINEARIZED EQUATIONS

Since $\mathbf{E}_1 \parallel \mathbf{B}_0 = B_0 \hat{z}$, $\mathbf{E}_1 = E_1 \hat{z}$ and let $\mathbf{k} = k\hat{x}$.

$$m_e \frac{\partial \mathbf{v}_{e1}}{\partial t} = -e\mathbf{E}_1 - e\mathbf{v}_{e1} \times \mathbf{B}_0 \longrightarrow \mathbf{v}_{e1} = \frac{e\mathbf{E}_1}{im_e\omega} : \quad \text{same as for } \mathbf{B}_0 = 0 \tag{4.166}$$

$$\mathbf{J}_1 = -en_0 \mathbf{v}_{e1} = \frac{in_0 e^2 \mathbf{E}_1}{m_e \omega} \tag{4.167}$$

$$\nabla(\nabla \cdot \mathbf{E}_1) - \nabla^2 \mathbf{E}_1 = -\frac{\omega_{pe}^2}{c^2}\mathbf{E}_1 - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_1}{\partial t^2} \tag{4.168}$$

DISPERSION RELATION

The wave equation becomes

$$k^2 \mathbf{E}_1 = \frac{\omega^2 - \omega_{pe}^2}{c^2} \mathbf{E}_1 \tag{4.169}$$

so the dispersion relation is given by

$$\boxed{\omega^2 = \omega_{pe}^2 + k^2 c^2} \tag{4.170}$$

so that the index of refraction is

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega^2}\,. \tag{4.171}$$

NOTES

- Same dispersion relation for $\mathbf{B}_0 = 0$.
  The ordinary wave propagates as if there were no magnetic field.
- Cut-off at $\omega = \omega_{pe}$.
- No propagation when $\omega < \omega_{pe}$.

4.11.2 Extraordinary Waves ($\mathbf{E}_1 \perp \mathbf{B}_0$)

ASSUMPTIONS:

Same as for the ordinary wave, but $\mathbf{E}_1 \perp \mathbf{B}_0$.

- Homogeneous infinite quasineutral plasma
- $\mathbf{E}_0 = 0$, $\mathbf{B}_0 \neq 0$
- Cold Plasma: $T_i = T_e = 0$
- Immobile ions: $\mathbf{v}_{i1} = 0$, $n_{i1} = 0$

FLUID EQUATIONS

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e\right] = -ne\mathbf{E} - en_e\mathbf{v}_e \times \mathbf{B} \tag{4.172}$$

$$\mathbf{J} = -en_0\mathbf{v}_e \tag{4.173}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.174}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \tag{4.175}$$

LINEARIZED EQUATIONS

Since $\mathbf{E}_1 \perp \mathbf{B}_0$, let $\mathbf{k} = k\hat{x}$ and $\mathbf{E}_1 = E_x\hat{x} + E_y\hat{y}$.

$$m_e\frac{\partial \mathbf{v}_{e1}}{\partial t} = -e\mathbf{E}_1 - e\mathbf{v}_{e1} \times \mathbf{B}_0 \tag{4.176}$$

$$\longrightarrow \begin{cases} i\omega v_x = \dfrac{e}{m_e}E_x + \omega_{ce}v_y \\ i\omega v_y = \dfrac{e}{m_e}E_y - \omega_{ce}v_x \end{cases}$$

$$\longrightarrow \begin{cases} v_x = \dfrac{e}{m_e\omega}\left(-iE_x - \dfrac{\omega_{ce}}{\omega}E_y\right)\left(1 - \dfrac{\omega_{ce}^2}{\omega^2}\right)^{-1} \\ v_y = \dfrac{e}{m_e\omega}\left(-iE_y + \dfrac{\omega_{ce}}{\omega}E_x\right)\left(1 - \dfrac{\omega_{ce}^2}{\omega^2}\right)^{-1} \end{cases}$$

$$\nabla(\nabla \cdot \mathbf{E}_1) - \nabla^2\mathbf{E}_1 = -\mu_0\frac{\partial \mathbf{J}_1}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}_1}{\partial t^2} \tag{4.177}$$

$$\longrightarrow (\omega^2 - k^2c^2)\mathbf{E}_1 + c^2kE_x\mathbf{k} = -\frac{i\omega}{\epsilon_0}\mathbf{J}_1 = \frac{in_0e\omega}{\epsilon_0}\mathbf{v}_{e1}$$

Or

$$\begin{aligned} \omega^2 E_x &= -\frac{i\omega n_0 e^2}{\epsilon_0 m\omega}\left(iE_x + \frac{\omega_{ce}}{\omega}E_y\right)\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)^{-1} \\ (\omega^2 - k^2c^2)E_y &= -\frac{i\omega n_0 e^2}{\epsilon_0 m\omega}\left(iE_y - \frac{\omega_{ce}}{\omega}E_x\right)\left(1 - \frac{\omega_{ce}^2}{\omega^2}\right)^{-1} \end{aligned}$$

Since $\omega_{pe}^2 = \frac{n_0 e^2}{m\epsilon_0}$,

$$\begin{aligned} \left[\omega^2\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)-\omega_{pe}^2\right]E_x + i\frac{\omega_{pe}^2\omega_{ce}}{\omega}E_y &= 0 \\ -i\frac{\omega_{pe}^2\omega_{ce}}{\omega}E_x + \left[(\omega^2-k^2c^2)\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)-\omega_{pe}^2\right]E_y &= 0 \end{aligned} \tag{4.178}$$

DISPERSION RELATION

$$\begin{vmatrix} \left[\omega^2\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)-\omega_{pe}^2\right] & i\frac{\omega_{pe}^2\omega_{ce}}{\omega} \\ -i\frac{\omega_{pe}^2\omega_{ce}}{\omega} & \left[(\omega^2-k^2c^2)\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)-\omega_{pe}^2\right] \end{vmatrix} = 0 \tag{4.179}$$

Or

$$\boxed{n^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega^2-\omega_{pe}^2}{\omega^2-\omega_{\mathrm{UH}}^2}} \tag{4.180}$$

NOTES

- Resonance: $n \longrightarrow \infty$, when $\omega^2 = \omega_{\mathrm{UH}}^2 = \omega_{pe}^2 + \omega_{ce}^2$.

  As a wave of given ω approaches the resonance point, $v_p \longrightarrow 0$ and $v_g \longrightarrow 0$, and the wave energy is converted into upper hybrid oscillation.

- Cutoff: $n = 0$

  $$1 = \frac{\omega_{pe}^2}{\omega^2}\frac{1}{1-\frac{\omega_{ce}^2}{\omega^2-\omega_{pe}^2}}$$

  Or

  $$\omega^2 \mp \omega_{ce}\omega - \omega_{pe}^2 = 0 \tag{4.181}$$

  so that $\omega = \omega_L, \omega_R$, where

  $$\begin{aligned} \omega_R &= \frac{1}{2}\left[+\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{pe}^2}\right] \\ \omega_L &= \frac{1}{2}\left[-\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{pe}^2}\right]. \end{aligned} \tag{4.182}$$

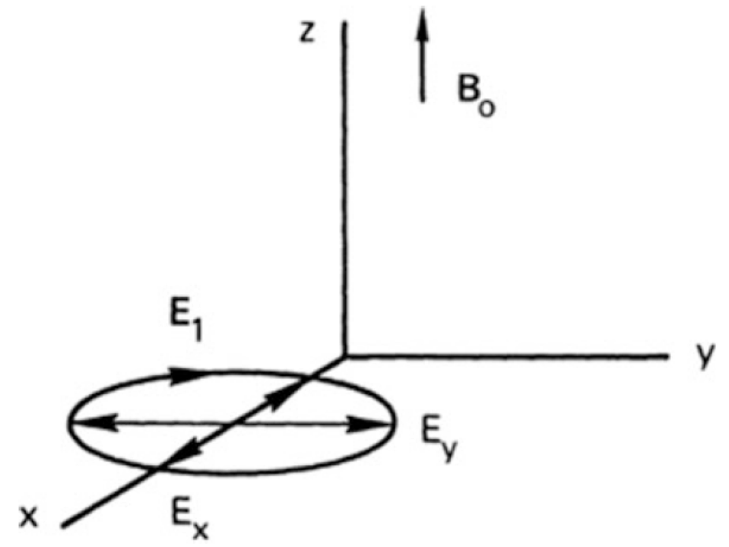

Figure 4.8: The E-vector of an extraordinary wave is elliptically polarized. The components E_x and E_y oscillate 90° out of phase, so that the total electric field vector E_1 has a tip that moves in an ellipse once in each wave period.

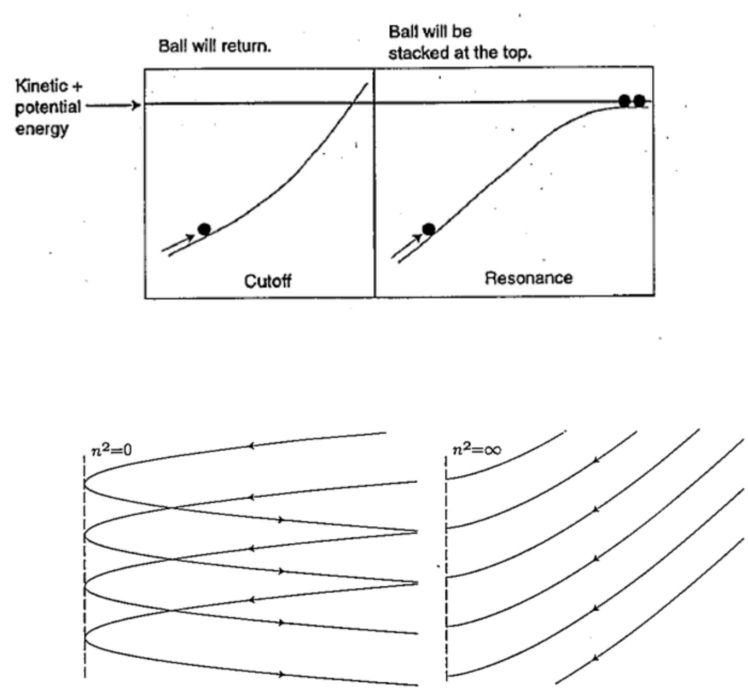

Figure 4.9: Mechanical analog to wave cutoffs and resonances (top). Behavior of the rays near cutoff and resonance surfaces (bottom).

4.12 Electromagnetic Waves with $\mathbf{k} \parallel \mathbf{B}_0$

ASSUMPTIONS

- Homogeneous infinite quasineutral plasma
- $\mathbf{E}_0 = 0$, $\mathbf{B}_0 \neq 0$
- Cold Plasma: $T_i = T_e = 0$
- Immobile ions: $\mathbf{v}_{i1} = 0$, $n_{i1} = 0$

FLUID EQUATIONS

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e\right] = -n_e e\mathbf{E} - en_e\mathbf{v}_e \times \mathbf{B} \tag{4.183}$$

$$\mathbf{J} = -en_0\mathbf{v}_e \tag{4.184}$$

$$\nabla \times (\nabla \times \mathbf{E}_1) = -\mu_0 \frac{\partial \mathbf{J}_1}{\partial t} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}_1}{\partial t^2} \tag{4.185}$$

LINEARIZED EQUATIONS

$\mathbf{k} = k\hat{z}$ and $\mathbf{E}_1 = E_x\hat{x} + E_y\hat{y}$.
The wave equation is

$$-\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1) + k^2\mathbf{E}_1 = i\omega\mu_0\mathbf{J}_1 + \frac{\omega^2}{c^2}\mathbf{E}_1\,, \tag{4.186}$$

or

$$(\omega^2 - k^2c^2)\mathbf{E}_1 = \frac{i\omega n_0 e}{\epsilon_0}\mathbf{v}_{e1}\,, \tag{4.187}$$

or

$$\begin{aligned}(\omega^2 - k^2c^2)E_x &= \frac{\omega_{pe}^2}{1 - \frac{\omega_{ce}^2}{\omega^2}}\left(E_x - i\frac{\omega_{ce}}{\omega}E_y\right),\\ (\omega^2 - k^2c^2)E_y &= \frac{\omega_{pe}^2}{1 - \frac{\omega_{ce}^2}{\omega^2}}\left(E_y + i\frac{\omega_{ce}}{\omega}E_x\right).\end{aligned} \tag{4.188}$$

Let

$$\alpha = \frac{\omega_{pe}^2}{1 - \frac{\omega_{ce}^2}{\omega^2}}\,, \tag{4.189}$$

then

$$\begin{aligned}(\omega^2 - k^2c^2 - \alpha)E_x + i\alpha\frac{\omega_{ce}}{\omega}E_y &= 0\\ -i\alpha\frac{\omega_{ce}}{\omega}E_x + (\omega^2 - k^2c^2 - \alpha)E_y &= 0\,.\end{aligned} \tag{4.190}$$

DISPERSION RELATION

$$\begin{vmatrix} (\omega^2 - k^2c^2 - \alpha) & i\alpha\dfrac{\omega_{ce}}{\omega} \\ -i\alpha\dfrac{\omega_{ce}}{\omega} & (\omega^2 - k^2c^2 - \alpha) \end{vmatrix} = 0 \tag{4.191}$$

Or

$$(\omega^2 - k^2c^2 - \alpha)^2 = (\alpha\frac{\omega_{ce}}{\omega})^2$$

$$\omega^2 - k^2c^2 - \alpha = \pm\alpha\frac{\omega_{ce}}{\omega} \tag{4.192}$$

Thus

$$\begin{aligned}\omega^2 - k^2c^2 &= \alpha\left(1 \pm \frac{\omega_{ce}}{\omega}\right) = \frac{\omega_{pe}^2}{1 - \frac{\omega_{ce}^2}{\omega^2}}\left(1 \pm \frac{\omega_{ce}}{\omega}\right) \\ &= \frac{\omega_{pe}^2\left(1 \pm \frac{\omega_{ce}}{\omega}\right)}{\left(1 + \frac{\omega_{ce}}{\omega}\right)\left(1 - \frac{\omega_{ce}}{\omega}\right)} = \frac{\omega_{pe}^2}{\left(1 \mp \frac{\omega_{ce}}{\omega}\right)}\end{aligned}$$

The index of refraction is given by

$$\boxed{n^2 = 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}}{\left(1 + \frac{\omega_{ce}}{\omega}\right)}} \quad \text{L-wave} \tag{4.193}$$

$$\boxed{n^2 = 1 - \frac{\frac{\omega_{pe}^2}{\omega^2}}{\left(1 - \frac{\omega_{ce}}{\omega}\right)}} \quad \text{R-wave} \tag{4.194}$$

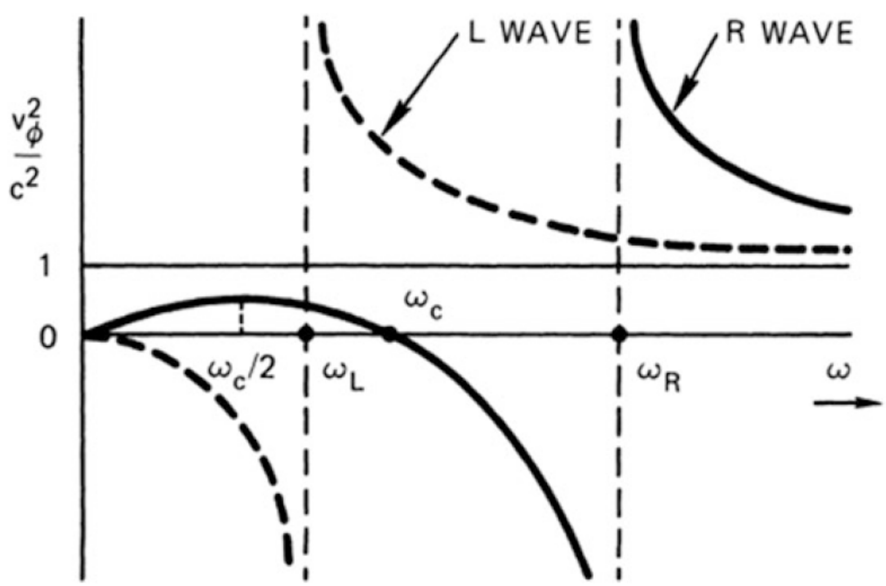

Figure 4.10: For $v_\phi^2/c^2 < 0$, they are regions of nonpropagation. The L wave has a stop band at low frequencies; the R wave has a stop band between ω_R and ω_c.

NOTES

- **Polarization for parallel propagation**
  We define the terms right-handed and left-handed in terms of the rotation of the electric field vector as a wave propagates.
  If the electric vector rotates clockwise as we look along the $\mathbf{k}$ direction, then this is a right-handed wave, and the left-handed wave rotates counterclockwise.
  To see what this implies, we consider a wave with complex E_x and E_y which represent a circularly polarized $R-$wave by representing

$$\begin{aligned} \mathrm{Re}\,[E_x] &= E\cos(-\omega t) &= \mathrm{Re}\,[Ee^{-i\omega t}] \\ \mathrm{Re}\,[E_y] &= -E\sin(-\omega t) &= \mathrm{Re}\,[iEe^{i\omega t}] \end{aligned} \tag{4.195}$$

  It is clear that the measurable field represented by Eq. (4.195) with real E rotates clockwise. Thus it follows that the phases of the waves are given by

$$\begin{aligned} iE_x &= E_y \qquad \text{R-wave} \\ iE_x &= -E_y \qquad \text{L-wave} \end{aligned} \tag{4.196}$$

- For upper sign in Eq. (4.192), which results in the dispersion relation represented by Eq. (4.194), we have $iE_x = E_y$ from Eq. (4.190).
  This verifies our labeling of the waves as being R-wave and L-wave.

- **R-wave**
  - Resonance at $\omega = \omega_{ce}$.
  - Cutoff at $\omega = \omega_R$.
  - The direction of rotation of the plane polarization is the same as the direction of gyration of electrons; the wave loses its energy in continuously accelerating the electrons, and it can not propagate.
  - **The whistler mode**
    For $\omega_{ci} \ll \omega \ll \omega_{ce} \sim \omega_{pe}$,

$$n^2 \simeq \frac{\omega_{pe}^2}{\omega\omega_{ce}} \tag{4.197}$$

    so that $k = \omega n/c = \frac{\omega_{pe}}{c}\sqrt{\omega/\omega_{ce}}$ or $\omega = k^2c^2\omega_{ce}/\omega_{pe}^2$ and the phase and group velocities are

$$\begin{aligned} v_p &= \frac{\omega}{k} = c\sqrt{\frac{\omega\omega_{ce}}{\omega_{pe}^2}} \propto \sqrt{\omega} \\ v_g &= \frac{d\omega}{dk} = \frac{2kc^2\omega_{ce}}{\omega_{pe}^2} = 2v_p \quad = 2c\sqrt{\frac{\omega\omega_{ce}}{\omega_{pe}^2}} \propto \sqrt{\omega}\,. \end{aligned} \tag{4.198}$$

    Note that both the phase and group velocities vary as $\sqrt{\omega}$ which causes high frequencies to propagate faster along the magnetic field lines.

- **L-wave**
  - Cutoff at $\omega = \omega_L$.
  - No resonance with the electrons.
    But if we had included ion motions, the L-wave would have a resonance at $\omega = \omega_{ci}$.

- **Faraday rotation**
  A linear polarized wave can be decomposed into a pair of right– and left– hand circularly polarized waves.

$$
\begin{aligned}
\mathbf{E}_R &= E e^{i(k_R z - \omega t)}(\hat{x} + i\hat{y}) \\
\mathbf{E}_L &= E e^{i(k_L z - \omega t)}(\hat{x} - i\hat{y})
\end{aligned}
\tag{4.199}
$$

$$
\mathbf{E}_{total} = \mathbf{E}_R + \mathbf{E}_L = E e^{-i\omega t}\left[\left(e^{ik_R z} + e^{ik_L z}\right)\hat{x} + i\left(e^{ik_R z} - e^{ik_L z}\right)\hat{y}\right] \tag{4.200}
$$

$$
\frac{E_y}{E_x} = i\frac{\left(e^{ik_R z} - e^{ik_L z}\right)}{\left(e^{ik_R z} + e^{ik_L z}\right)} = i\frac{1 - e^{i(k_L - k_R)z}}{1 + e^{i(k_L - k_R)z}} = \tan\left[\frac{1}{2}(k_L - k_R)z\right] \tag{4.201}
$$

  A plane-polarized wave sent along a magnetic field in a plasma suffers a rotation of its plane of rotation. Faraday rotation can be used as a diagnostic for estimating plasma densities in laboratory plasma and interstellar space.

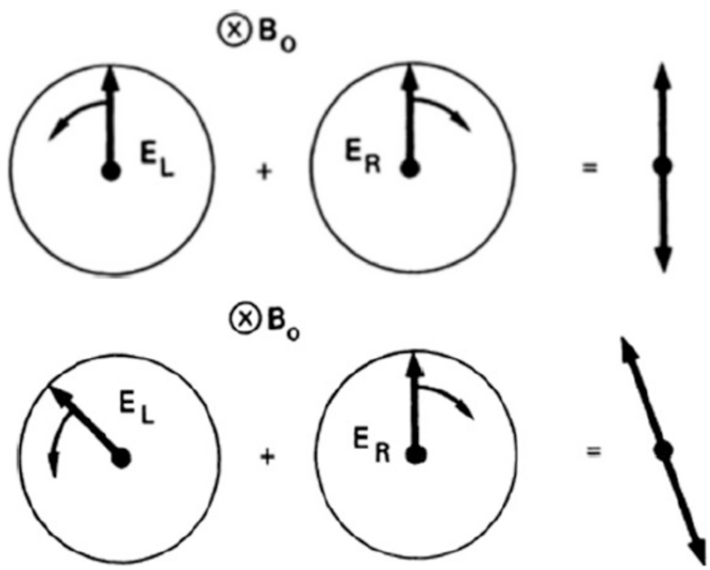

Figure 4.11: A plane-polarized wave as the sum of left and righthanded circularly polarized waves (top). After traversing the plasma, the L wave is advanced in phase relative to the R wave, and the plane of polarization is rotated (bottom).