4.5 Ion Acoustic Waves (Ion Plasma Waves)

In the absence of collisions, ordinary sound waves would not occur.
But acoustic waves can occur through the intermediary of an electric field.

ASSUMPTIONS

- Homogeneous infinite plasma
- No external field: $\mathbf{E}_0 = \mathbf{B}_0 = 0$
- Warm Plasma: $T_i \neq 0$, $T_e \neq 0$
- Moving ions: $\mathbf{v}_{i1} \neq 0$ $n_{i1} \neq 0$
- Electrostatic field: $\mathbf{E} = -\nabla\phi$ (same as in electron plasma waves).
- The plasma approximation ($n_i = n_e$ but finite $\mathbf{E}$).

FLUID EQUATIONS

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0 \tag{4.76}$$

$$m_i n_i \left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i\right] = e n_i \mathbf{E} - \nabla p_i = -e n_i \nabla\phi - \gamma_i K T_i \nabla n_i \tag{4.77}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{4.78}$$

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e\right] = -e n_e \mathbf{E} - \nabla p_e = +e n_e \nabla\phi - \gamma_e K T_e \nabla n_e \tag{4.79}$$

LINEARIZED EQUATIONS

$$\frac{\partial n_{i1}}{\partial t} + n_0 \nabla \cdot \mathbf{v}_{i1} = 0 \tag{4.80}$$

$$m_i n_0 \frac{\partial \mathbf{v}_{i1}}{\partial t} = -e n_0 \nabla\phi_1 - \gamma_i K T_i \nabla n_{i1} \tag{4.81}$$

$$\frac{\partial n_{e1}}{\partial t} + n_0 \nabla \cdot \mathbf{v}_{e1} = 0 \tag{4.82}$$

$$m_e n_0 \frac{\partial \mathbf{v}_{e1}}{\partial t} = +e n_0 \nabla\phi_1 - \gamma_e K T_e \nabla n_{e1} \tag{4.83}$$

The continuity equations become

$$-i\omega n_{i1} + n_0 i k v_{i1} = 0 \tag{4.84}$$

$$-i\omega n_{e1} + n_0 i k v_{e1} = 0 \tag{4.85}$$

From these equations, we obtain $v_{e1} = v_{i1}$ (since $n_{e1} = n_{i1}$).
The momentum equations become

$$-i\omega n_0 m_i v_{i1} = ik(-en_0\phi_1 - \gamma_i KT_i n_{i1}) \tag{4.86}$$

$$-i\omega n_0 m_e v_{e1} = ik(+en_0\phi_1 - \gamma_e KT_e n_{e1}) \tag{4.87}$$

Add these two equations to get

$$-i\omega n_0(m_i + m_e)v_{i1} = -ik(\gamma_e KT_e + \gamma_i KT_i)n_{i1} \tag{4.88}$$

which can be solved simultaneously with Eq. (4.84).

DISPERSION RELATION

$$\begin{vmatrix} k(\gamma_e KT_e + \gamma_i KT_i) & n_0\omega(m_i + m_e) \\ -\omega & n_0 k \end{vmatrix} = 0 \tag{4.89}$$

Or

$$\frac{\omega^2}{k^2} = \frac{\gamma_e KT_e + \gamma_i KT_i}{m_i + m_e} \tag{4.90}$$

Neglecting m_e,

$$\boxed{\frac{\omega}{k} = \sqrt{\frac{\gamma_e KT_e + \gamma_i KT_i}{m_i}} = v_s} \tag{4.91}$$

NOTES

- Phase velocity: $v_\phi = \frac{\omega}{k} = v_s$
- Group velocity: $v_g = \frac{d\omega}{dk} = v_s$
- A typical electron travels many wavelength in one wave period; that is, the distance traveled in one period

  $$v_e/\omega \sim v_e/kv_s \gg k^{-1} \sim \lambda$$

  since

  $$\frac{v_e}{v_s} \sim \sqrt{\frac{KT_e}{m_e}\frac{m_i}{\gamma_e KT_e + \gamma_i KT_i}} \sim \sqrt{\frac{m_i}{m_e}} \gg 1$$

  Thus the electrons are communicating over many wavelengths during one wave period so that they remain isothermal. So we may take $\gamma_e = 1$.
- For ions,

  $$v_i/\omega \sim v_i/kv_s < k^{-1} \sim \lambda$$

  Thus a typical ion travels only a fraction of a wavelength λ in one wave period; then the compression of the wave may be an adiabatic one.
- Using the kinetic theory, it can be shown that the wave is heavily damped unless $T_i \ll T_e$. We may interpret this as follows:

– When the electron and ion temperatures are comparable, there is generally no oscillation, since when one moves some of the ions to form a slight charge imbalance which would induces the ions to oscillate, the period is so slow that electrons, being so much more mobile, rush in to fill the charge imbalance before the ions can respond.
– When the electron temperature is much higher than the ion temperature, the electrons have too much momentum to slow down and fill in the potential depression so some ion oscillates may occur.

If $T_i \ll T_e$, $v_s = \sqrt{KT_e/m_i}$, which depends on electron temperature and on ion mass (the fluid's inertia is proportional to it). The speed is the thermal speed that the ions would have if they had the electron temperature. Note that ion acoustic waves still exist even if the ion temperature goes to zero unlike ordinary sound waves.

4.6 The Validity of the Plasma Approximation

In the preceding analysis, we used the plasma approximation; we took $n_{e1} = n_{i1}$ and did not use Poission's equation.
How accurate is this approximation?

Neglecting the electron inertia leads to the Boltzmann relation from the momentum equation. Thus we have for the electrons

$$n_{e1} = \frac{e\phi_1}{KT_e} \,. \tag{4.92}$$

Poisson's equation $\epsilon_0 \nabla \cdot \mathbf{E}_1 = e(n_{i1} - n_{e1})$ yields

$$\epsilon_0 k^2 \phi_1 = e\left(n_{i1} - \frac{en_0}{KT_e}\phi_1\right) \tag{4.93}$$

so that we obtain

$$\epsilon_0 \left(k^2 + \frac{e^2 n_0}{\epsilon_0 KT_e}\right)\phi_1 = en_{i1} \tag{4.94}$$

or

$$\epsilon_0 \left(k^2 \lambda_D^2 + 1\right)\phi_1 = en_{i1}\lambda_D^2 \,. \tag{4.95}$$

From the ion continuity equation, we get

$$-i\omega n_{i1} + n_0 ikv_{i1} = 0 \longrightarrow n_0 v_{i1} = \frac{\omega}{k} n_{i1} \,. \tag{4.96}$$

The ion momentum equation is

$$-i\omega m_i n_0 v_{i1} = ik(-en_0\phi_1 - \gamma_i KT_i n_{i1}) \tag{4.97}$$

which is then rewritten

$$\frac{\omega^2}{k^2} n_{i1} = \left[\frac{e^2 n_0 \lambda_D^2}{m_i \epsilon_0 (k^2\lambda_D^2 + 1)} + \frac{\gamma_i KT_i}{m_i}\right] n_{i1} \tag{4.98}$$

so that the dispersion relation is found to be

$$\boxed{\frac{\omega^2}{k^2} = \frac{KT_e}{m_i}\frac{1}{1+k^2\lambda_D^2} + \frac{\gamma_i KT_i}{m_i}}. \tag{4.99}$$

NOTES

- When $k^2\lambda_D^2 = 4\pi^2(\frac{\lambda_D}{\lambda})^2 \ll 1$ (the wavelength is very large compared to the Debye length),

  $$\frac{\omega^2}{k^2} = \left[\frac{KT_e}{m_i} + \frac{\gamma_i KT_i}{m_i}\right] \tag{4.100}$$

  This is the same result when the plasma approximation is used.
  For long wavelength, the plasma approximation is valid.

- When $k^2\lambda_D^2 \gg 1$ and $T_i \longrightarrow 0$,

  $$\omega^2 = k^2\frac{n_0e^2}{\epsilon_0 m_i k^2} = \frac{n_0e^2}{m_i\epsilon_0} = \omega_{pi}^2 \tag{4.101}$$

  where ω_{pi} is the ion plasma frequency.
  For short wavelength (large k), the ion acoustic wave turns into a constant-frequency wave. Because the wavelength is short compared to the Debye length, the electrons are incapable of shielding, and we have ions oscillating in a uniform background of negative charge.

- - The electron plasma wave is basically constant frequency, but become constant velocity wave at large k.
  - The ion acoustic wave is basically constant velocity wave, but becomes constant frequency wave at large k.

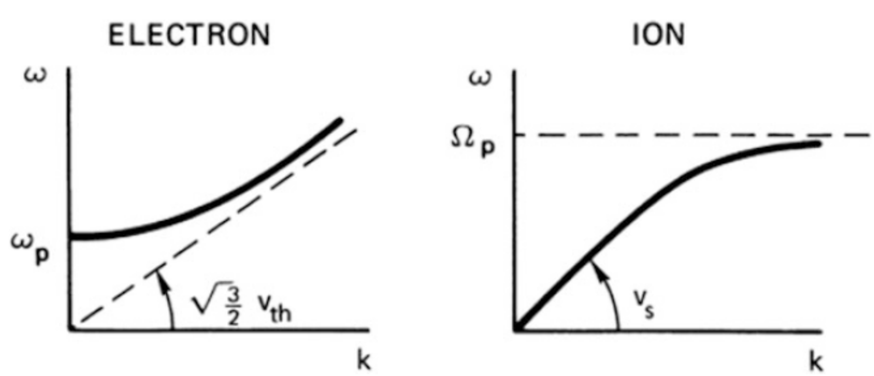

Figure 4.5:

4.7 Electrostatic Electron Oscillations with $\mathbf{B}_0 \neq 0$

ASSUMPTIONS

Same as that of plasma oscillation except $\mathbf{B}_0 \neq 0$

- Homogenious infinite plasma
- $\mathbf{E}_0 = 0$, but $\mathbf{B}_0 \neq 0$
- Cold Plasma: $T_i = 0$, $T_e = 0$
- Fixed ions: $\mathbf{v}_{i1} = 0$, $n_{i1} = 0$
- Electrostatic field: $\nabla \times \mathbf{E}_1 = 0$

FLUID EQUATIONS

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{4.102}$$

$$m n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e \right] = -n_e e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}_0) \tag{4.103}$$

$$\nabla \cdot \mathbf{E} = \frac{e}{\epsilon_0}(n_i - n_e) \tag{4.104}$$

LINEARIZED EQUATIONS

$$\frac{\partial n_{e1}}{\partial t} + n_0 \nabla \cdot \mathbf{v}_{e1} = 0 \tag{4.105}$$

$$m \frac{\partial \mathbf{v}_{e1}}{\partial t} = -e(\mathbf{E}_1 + \mathbf{v}_{e1} \times \mathbf{B}_0) \tag{4.106}$$

$$\nabla \cdot \mathbf{E}_1 = -\frac{e}{\epsilon_0} n_{e1} \tag{4.107}$$

DISPERSION RELATION

- When $\mathbf{k} \parallel \mathbf{B}_0$
  Let $\mathbf{k} = k\hat{z}$ ($\mathbf{E} = E\hat{z}$) and $\mathbf{B}_0 = B_0\hat{z}$.
  Then the linearized perturbation equations become

$$\begin{cases} -i\omega m v_x = -e v_y B_0 \\ -i\omega m v_y = +e v_x B_0 \\ -i\omega m v_z = -eE \longrightarrow v_z = \dfrac{eE}{im\omega} \end{cases} \tag{4.108}$$

$$-\omega n_1 + n_0 k v_z = 0 \longrightarrow n_1 = \frac{k}{\omega} n_0 v_z \tag{4.109}$$

$$ikE + \frac{e}{\epsilon_0} n_1 = 0 \longrightarrow \left(\frac{n_0 e^2}{\epsilon_0 m \omega^2} - 1 \right) E = 0 \tag{4.110}$$

Hence, we obtain

$$\boxed{\omega = \omega_{pe}} \quad \text{Same for the case of } \mathbf{B}_0 = 0\,. \tag{4.111}$$

- When $\mathbf{k} \perp \mathbf{B}_0$
  Let $\mathbf{k} = k\hat{x}$ ($\mathbf{E} = E\hat{x}$) and $\mathbf{B}_0 = B_0\hat{z}$.
  Then the linearized perturbation equations become

$$\left.\begin{aligned} -i\omega m v_x &= -eE \quad -ev_y B_0 \\ -i\omega m v_y &= \qquad\quad +ev_x B_0 \end{aligned}\right\} \longrightarrow v_x = \frac{eE/im\omega}{1 - \omega_{ce}^2/\omega^2} \tag{4.112}$$

$$-\omega n_1 + n_0 k v_x = 0 \longrightarrow n_1 = \frac{k}{\omega} n_0 v_x \tag{4.113}$$

$$ikE + \frac{e}{\epsilon_0} n_1 = 0 \longrightarrow \left(1 - \frac{\omega_{ce}^2}{\omega^2} \right) E = \frac{\omega_{pe}^2}{\omega^2} E \tag{4.114}$$

so that the dispersion relation is

$$\boxed{\omega = \sqrt{\omega_{pe}^2 + \omega_{ce}^2} \equiv \omega_{\mathrm{UH}}} \quad \text{upper hybrid frequency} \tag{4.115}$$

NOTES

- $v_g = 0$: No propagation
- When $\mathbf{k} \parallel \mathbf{B}_0$, $\omega = \omega_{pe}$: The magnetic field does not affect the parallel motion.
- When $\mathbf{k} \perp \mathbf{B}_0$, $\omega^2 = \omega_{pe}^2 + \omega_{ce}^2$
  Note that
  $\mathbf{B}_0 \longrightarrow 0$: $\quad \omega \longrightarrow \omega_{pe}$
  $n \longrightarrow 0$: $\quad \omega \longrightarrow \omega_{ce}$
  Because the Lorentz force acts as an extra restoring force for the oscillation, the frequency is higher than the plasma oscillation where the restoring force is the electrostatic field. As the magnetic field goes to zero, one recovers a plasma oscillation. As the plasma density goes to zero, one has a simple Larmour gyration, since the electrostatic forces vanish with density.

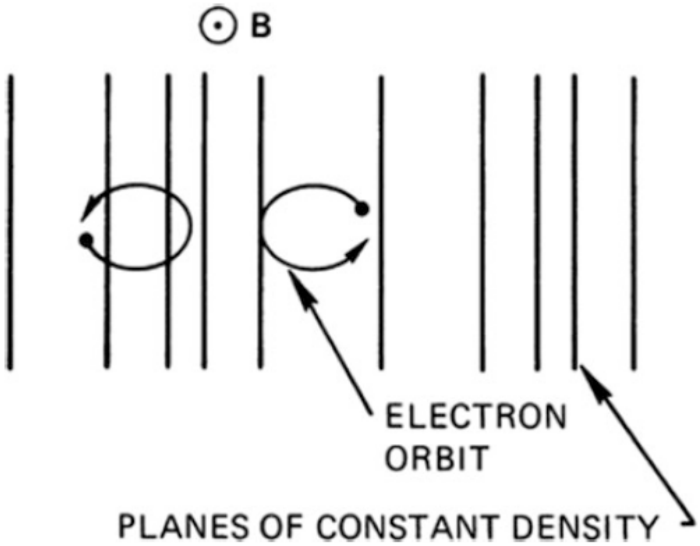

Figure 4.6:

4.8 Electrostatic Ion Waves with $\mathbf{B}_0 \neq 0$

ASSUMPTIONS:

Same as that of ion waves except $\mathbf{B}_0 \neq 0$

- Homogeneous infinite plasma
- $\mathbf{E}_0 = 0$, $\mathbf{B}_0 \neq 0$
- Warm Plasma: $T_i \neq 0$, $T_e \neq 0$
- Moving ions: $\mathbf{v}_{i1} \neq 0$, $n_{i1} \neq 0$
- Plasma approximation ($n_{i1} = n_{e1} \equiv n_1$)
- $\mathbf{k} = k_x\hat{x} + k_z\hat{z}$

FLUID EQUATIONS

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0 \tag{4.116}$$

$$m_i n_i \left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i\right] = en_i\mathbf{E} - \nabla p + en_i\mathbf{v}_i \times \mathbf{B}_0 \tag{4.117}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{4.118}$$

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e\right] = en_e\mathbf{E} - \nabla p - en_e\mathbf{v}_e \times \mathbf{B}_0 \tag{4.119}$$

LINEARIZED EQUATIONS

$$i\omega n_1 + n_0 i\mathbf{k}\cdot\mathbf{v}_i = 0 \tag{4.120}$$

$$-m_i n_0 i\omega\mathbf{v}_i = en_0 E + en_0\mathbf{v}_1\times\mathbf{B}_0 - i\mathbf{k}n_1\gamma_i KT_i \tag{4.121}$$

$$-i\omega n_1 + n_0 i\mathbf{k}\cdot\mathbf{v}_e = 0 \tag{4.122}$$

$$-m_e n_0 i\omega\mathbf{v}_e = -en_0 E - en_0\mathbf{v}_e\times\mathbf{B}_0 - i\mathbf{k}n_1\gamma_e KT_e \tag{4.123}$$

DISPERSION RELATION

Add the momentum equations for each speci to get

$$-i\omega n_0(m_e\mathbf{v}_e + m_i\mathbf{v}_i) = en_0(\mathbf{v}_i - \mathbf{v}_e)\times\mathbf{B}_0 - i\mathbf{k}(\gamma_i KT_i + \gamma_e KT_e)n_1\,. \tag{4.124}$$

Take the dot product this equation with $\mathbf{k}$ to obtain

$$-i\omega n_0(m_e\mathbf{k}\cdot\mathbf{v}_e + m_i\mathbf{k}\cdot\mathbf{v}_i) = -ik^2(\gamma_i KT_i + \gamma_e KT_e)n_1 + en_0\mathbf{k}\cdot[(\mathbf{v}_i - \mathbf{v}_e)\times\mathbf{B}_0]\,. \tag{4.125}$$

Use the continuity equations $\mathbf{k}\cdot\mathbf{v}_{e,i} = \frac{n_1}{n_0}\omega$ to get

$$-i\omega^2 n_1 = -ik^2 v_s^2 n_1 + \omega_{ci} n_0 k_x(v_{iy} - v_{ey})\,. \tag{4.126}$$

To find the expression between v_y and n_1, we take the cross product of the momentum equations with $\mathbf{k}$:

$$-i\omega m\mathbf{k}\times\mathbf{v} = \pm e\mathbf{k}\times(\mathbf{v}\times\mathbf{B}_0) \tag{4.127}$$

with $+$ for the ion and $-$ for the electron, which gives

$$\begin{aligned} -k_z v_y &= \pm i\tfrac{\omega_c}{\omega}k_z v_x \\ k_z v_x - k_x v_z &= \pm i\tfrac{\omega_c}{\omega}k_z v_y \\ k_x v_y &= \pm i\tfrac{\omega_c}{\omega}(-k_x v_x) \end{aligned} \tag{4.128}$$

These can be solved for v_x and v_z in terms of v_y,

$$\begin{aligned} v_x &= \alpha v_y \\ v_z &= \frac{k_z}{k_x}\alpha\left(1 + \frac{1}{\alpha^2}\right)v_y \end{aligned} \tag{4.129}$$

where

$$-\alpha^{-1} = \pm i\frac{\omega_c}{\omega}$$

From the continuity equation,

$$k_x v_x + k_z v_z = \frac{n_1}{n_0}\omega \tag{4.130}$$

we obtain

$$\alpha\left[k_x + \frac{k_z^2}{k_x}\left(1+\frac{1}{\alpha^2}\right)\right] v_y = \frac{n_1}{n_0}\omega \tag{4.131}$$

or

$$n_0 v_y = \frac{\mp i\omega_c n_1}{k_x\left[1+\frac{k_z^2}{k_x^2}\left(1-\frac{\omega_c^2}{\omega^2}\right)\right]} \tag{4.132}$$

Finally the dispersion relation is found to be

$$\omega^2 = k^2 v_s^2 + \left[\frac{\omega_{ci}^2}{1+\frac{k_z^2}{k_x^2}\left(1-\frac{\omega_{ci}^2}{\omega^2}\right)} + \frac{\omega_{ci}\omega_{ce}}{1+\frac{k_z^2}{k_x^2}\left(1-\frac{\omega_{ce}^2}{\omega^2}\right)}\right] \tag{4.133}$$

NOTES

- When $k_x = 0$ ($\mathbf{k} \parallel \mathbf{B}_0$),

  $$\boxed{\omega^2 = k^2 v_s^2} \quad \text{ion acoustic wave}\,. \tag{4.134}$$

  Notice again that the magnetic field does not influence the wave properties in the parallel direction.

- When $k_z = 0$ ($\mathbf{k} \perp \mathbf{B}_0$),

  $$\boxed{\omega^2 = k^2 v_s^2 + \omega_{\rm LH}^2} \quad \text{lower hybrid wave} \tag{4.135}$$

  where

  $$\boxed{\omega_{\rm LH} = \sqrt{\omega_{ci}\omega_{ce}}} \quad \text{lower hybrid frequency}\,. \tag{4.136}$$

  For cold plasmas ($T_e = T_i = 0$), $v_s = 0$. Then

  $$\omega = \omega_{LH}: \quad \text{lower hybrid oscillation} \tag{4.137}$$

- When $\sqrt{\frac{m_e}{m_i}} \ll \left|\frac{k_z}{k_x}\right| \ll 1$ and $\omega \sim \omega_{ci}$, the dispersion relation is

  $$\boxed{\omega^2 = k^2 v_s^2 + \omega_{ci}^2} \quad \text{electrostatic ion cyclotron wave}\,. \tag{4.138}$$