Chapter 3

PLASMAS AS FLUIDS

3.1 Introduction

Single charged particle motion:

- Prescribed **E** and **B** $\longrightarrow$ Eq. of motion $\longrightarrow$ **r** and **v**

In a plasma, the **E** and **B** fields are not prescribed but are determined by the position and motions of the charges themselves.
One must solve a self-consistent problem.

Plasma Motions:

- Fluid Theory:
    1. The identity of the individual particle is neglected.
    2. Only the motion of fluid elements are taken into account.
    3. One deals with the macroscopic variables averaged over a distribution function.
- Kinetic Theory:
    1. The identity of the individual particle is also neglected.
    2. But the velocity distribution is considered.

3.2 The Fluid Equation of Motion

3.2.1 Macroscopic Variables of a Plasma

Observable properties of a system of particles are obtained through quantities averaged over a distribution function. The average of a physical quantity $M(\mathbf{r}, \mathbf{v}, t)$ is defined by

$$\langle M(\mathbf{r}, \mathbf{v}, t)\rangle = \frac{\int M f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}}{\int f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}} \,. \tag{3.1}$$

- Particle number density:

$$n(\mathbf{r}, t) = \int_{-\infty}^{\infty} f(\mathbf{r}, \mathbf{v}, t) d\mathbf{v} \tag{3.2}$$

- Average velocity:

$$\mathbf{u}(\mathbf{r},t) = \langle \mathbf{v} \rangle = \frac{1}{n}\int_{-\infty}^{\infty} \mathbf{v} f(\mathbf{r},\mathbf{v},t) d\mathbf{v} \tag{3.3}$$

- Pressure tensor:

$$\mathsf{P}(\mathbf{r},t) = \frac{1}{n}\int mn(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u}) f(\mathbf{r},\mathbf{v},t) d\mathbf{v} \tag{3.4}$$

  In index form

$$\begin{aligned}
\mathsf{P}_{ij} &= mn \left\langle (v_i - u_i)(v_j - u_j) \right\rangle \\
&= mn \left(\langle v_i v_j \rangle - u_i u_j \right) \\
&= p_{ii}\delta_{ij} + \Pi_{ij} \\
&= \text{scalar pressure} + \text{shear stress} \\
&= \text{flux in the } i\text{-direction of } j\text{-directed momentum}
\end{aligned}$$

  For an isotropic velocity distribution (e.g., Maxwellian) this reduces to a diagonal pressure tensor

$$\mathsf{P} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \tag{3.5}$$

  and p is the scalar pressure (assuming $T_i = T_\perp = T_\parallel$)

$$p = nKT\,. \tag{3.6}$$

  We note for an isotropic velocity distribution,

$$f(\mathbf{r},\mathbf{v},t) = f(\mathbf{r},|\mathbf{v}-\mathbf{u}|,t) \tag{3.7}$$

  so for $i = j$,

$$p_{ii} = \frac{1}{n}\int mn(\mathbf{v}-\mathbf{u})_i^2 f d\mathbf{v} = 2n\left(\frac{1}{n}\int \frac{1}{2}m(\mathbf{v}-\mathbf{u})_i^2 f d\mathbf{v}\right) = 2n \times \frac{1}{2}KT \tag{3.8}$$

  and for $i \neq j$ (due to cancellation between odd functions)

$$p_{ij} = 0 \tag{3.9}$$

- Charge density:

$$\rho(\mathbf{r},t) = \sum_s \int q_s f_s d\mathbf{v} = \sum_s q_s n_s(\mathbf{r},t) \tag{3.10}$$

  where the summation is carried over all particle species.

- Current density:

$$\mathbf{J}(\mathbf{r},t) = \sum_s \int q_s \mathbf{v} f_s d\mathbf{v} = \sum_s q_s n_s(\mathbf{r},t)\mathbf{u}_s(\mathbf{r},t) \tag{3.11}$$

  where again the summation is carried over all particle species.

3.2.2 Equation of Continuity

The conservation of the total particle number over a volume V requires

$$\frac{\partial N}{\partial t} = \int_V \frac{\partial n}{\partial t} dv = -\oint n\mathbf{u} \cdot d\mathbf{a} = -\int_V \nabla \cdot (n\mathbf{u}) dv \tag{3.12}$$

or

$$\boxed{\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0}\,. \tag{3.13}$$

3.2.3 Equation of State

The equation of state (for an ideal gas of ν moles. $\nu = N/N_A$ and $K = R/N_A$):

$$pV = \nu RT \tag{3.14}$$

or

$$\boxed{p = nKT}\,. \tag{3.15}$$

The first law of thermodynamics states

$$dQ = dU + dW \tag{3.16}$$

where U is the mean internal energy of the gas. The specific heat of the gas per mole is defined as

$$C = \frac{1}{\nu}\frac{dQ}{dT}\,. \tag{3.17}$$

- At constant volume, $dW = pdV = 0$ so that we have

$$dQ = \nu C_v dT = dU\,. \tag{3.18}$$

- At constant pressure, $dQ = \nu C_p dT$ by definition.
  Thus for the constant pressure process the first law yields

$$\nu C_p dT = \nu C_V dT + pdV\,. \tag{3.19}$$

  Since

$$pdV = \nu RdT \tag{3.20}$$

  we find

$$C_p - C_V = R\,. \tag{3.21}$$

- For an ideal gas undergoing an adiabatic process ($dQ = 0$),

$$\nu C_v dT + pdV = 0 \tag{3.22}$$

  or

$$dT = -\frac{pdV}{\nu C_V} \tag{3.23}$$

From the equation of state, $pdV + Vdp = \nu RdT$ so that we have

$$dT = \frac{pdV + Vdp}{\nu R}\,. \tag{3.24}$$

Equating these two expressions and using $C_p - C_v = R$, we obtain

$$C_p pdV + C_V Vdp = 0 \tag{3.25}$$

or

$$\frac{dp}{p} + \gamma\frac{dV}{V} = 0 \tag{3.26}$$

where

$$\gamma = \frac{C_p}{C_V} = \frac{2+f}{f} \tag{3.27}$$

Thus

$$\ln p + \gamma \ln V = const \tag{3.28}$$

or

$$\boxed{pV^\gamma = const}\,. \tag{3.29}$$

- For an isothermal process ($T = const$),

$$\boxed{pV = const}\,. \tag{3.30}$$

NOTES:

- For an adiabatic process (compression is faster than thermal conduction),

$$pV^\gamma = const \tag{3.31}$$

or

$$p\rho_m^{-\gamma} = const \equiv c \tag{3.32}$$

$$p = c\rho_m^\gamma = c(mn)^\gamma\,. \tag{3.33}$$

Therefore we have

$$\nabla p = \gamma c(mn)^{\gamma-1} m\nabla n \tag{3.34}$$

or

$$\frac{\nabla p}{p} = \gamma\frac{\nabla n}{n}\,. \tag{3.35}$$

or

$$\nabla p = \gamma KT\nabla n \tag{3.36}$$

- For an isothermal process (compression is slow compared to thermal conduction),

$$p = nKT \tag{3.37}$$

thus

$$\nabla p = kT\nabla n\,. \tag{3.38}$$

3.2.4 Equation of Motion

Convective Derivative (material derivative or Lagrangian derivative)
Change of physical quantity $\mathbf{G}$ in moving frame is

$$\frac{d}{dt}\mathbf{G}(\mathbf{r},t) = \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \mathbf{G}}{\partial x}\frac{dx}{dt} + \cdots = \frac{\partial \mathbf{G}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{G}$$

- partial derivative: $\frac{\partial}{\partial t} \longrightarrow$ change with time at a fixed position in space
- convective derivative: $\frac{d}{dt} \longrightarrow$ derivative following the motion

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla = \frac{D}{Dt} \tag{3.39}$$

where $\mathbf{u}$ is the local fluid velocity. For a plasma, take $\mathbf{G}$ to be the fluid velocity $\mathbf{u}$ to get

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = nq(\mathbf{E} + \mathbf{u}\times\mathbf{B})\,. \tag{3.40}$$

Pressure
Let us consider some volume in a fluid. The total force acting on this volume is equal to the integral

$$-\oint_S p d\mathbf{a} \tag{3.41}$$

of the pressure, taken over the surface bounding the volume. Transforming it to a volume integral, we have

$$-\oint_S p d\mathbf{a} = -\int_V \nabla p dv \tag{3.42}$$

We can say that a force $-\nabla p$ acts on unit volume of the fluid. Thus the fluid equation of motion is

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = nq(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nabla p \tag{3.43}$$

Collision
The momentum loss per collision is proportional to the relative velocity $\mathbf{u} - \mathbf{u}_0$, where $\mathbf{u}_0$ is the velocity of the neutral fluid.
If τ, the mean free time between collisions, is approximately constant, the resulting force term can be roughly written as

$$\frac{\Delta \mathbf{p}}{\Delta t} = -\frac{mn(\mathbf{u} - \mathbf{u}_0)}{\tau} \tag{3.44}$$

or

$$-mn\nu(\mathbf{u} - \mathbf{u}_0) \tag{3.45}$$

where $\nu = 1/\tau$ (mean collision frequency). Therefore, the fluid equation of motion is

$$\boxed{mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = nq(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nabla p - mn\nu(\mathbf{u} - \mathbf{u}_0)}\,. \tag{3.46}$$

3.2.5 The Complete Set of Fluid Equation (with $\mathbf{u} \to \mathbf{v}$)

The charge and current densities are given by

$$\rho = n_i q_i + n_e q_e \tag{3.47}$$

$$\mathbf{J} = n_i q_i \mathbf{v}_i + n_e q_e \mathbf{v}_e \tag{3.48}$$

Assuming

- Isotropic: $-\nabla \cdot \mathsf{P} = -\nabla p \mathsf{I} = -\nabla p$ (pressure only)
- No collisions: $-mn\nu(\mathbf{u} - \mathbf{u}_0) = 0$
- No external sources: $\rho_{ext} = \mathbf{J}_{ext} = 0$,

we have

$$\epsilon_0 \nabla \cdot \mathbf{E} = n_i q_i + n_e q_e \tag{3.49}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.50}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.51}$$

$$\frac{1}{\mu_0} \nabla \times \mathbf{B} = n_i q_i \mathbf{v}_i + n_e q_e \mathbf{v}_e + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{3.52}$$

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0 \tag{3.53}$$

$$m_s n_s \left[\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s \right] = -\nabla p_s + n_s q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) \tag{3.54}$$

$$p_s = C_s n_s^{\gamma_s} \tag{3.55}$$

for $s = i, e$.

NOTES:

- $\mathbf{E}$, $\mathbf{B}$, n_i, n_e, $\mathbf{v}_i$, $\mathbf{v}_e$, p_i, p_e: 16 scalar unknowns.
- 18 scalar equations with two redundant equations. Note that two divergence equations are superfluous since they can be reduced from curl equations.
- The simultaneous solution of this set of 16 equations in 16 unknowns gives a self consistent set of fields and motions in the fluid approximation.
- In plasma physics, one generally works with the vacuum Maxwell equations (i.e., no bound charges and currents), and includes all the charges and currents, both external and internal (i.e., from the plasma).
- Each time we take a moment of Vlasov equation, an equation for the moment is obtained, but at the same time, a next higher moment also appears. Thus moment-taking never leads to a closed system of equation. Some sort of ad hoc closure must always be invoked to terminate this chain. Typical closures involve invoking adiabatic or isothermal assumptions. (From Bellan)

3.3 Fluid Drifts Perpendicular to B

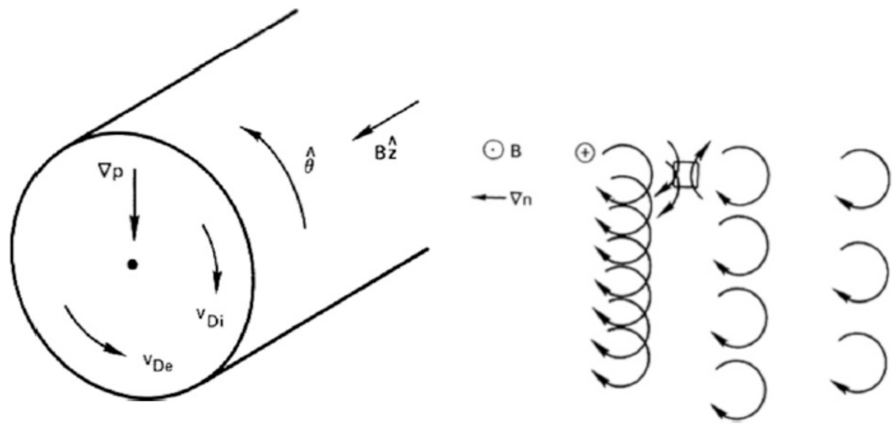

Figure 3.1:

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = nq(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - \nabla p \tag{3.56}$$

Consider the ratio of term (1) to (3) ($\frac{\partial}{\partial t} \longrightarrow i\omega$)

$$\frac{(1)}{(3)} \sim \left|\frac{mni\omega v}{qnvB}\right| \sim \frac{\omega}{\omega_c} \ll 1 \quad \text{for low frequency motion}$$

If the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term is neglected, the equation of motion is reduced to

$$0 = nq(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - \nabla p \tag{3.57}$$

Take the cross product of Eq. (3.57) with $\mathbf{B}$ to get

$$\begin{aligned} 0 &= -\nabla p\times\mathbf{B} + nq[\mathbf{E}\times\mathbf{B} + (\mathbf{v}_\perp\times\mathbf{B})\times\mathbf{B}] \\ &= -\nabla p\times\mathbf{B} + nq\mathbf{E}\times\mathbf{B} - nq\mathbf{v}_\perp B^2\,. \end{aligned}$$

So we have

$$\mathbf{v}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2} - \frac{\nabla p\times\mathbf{B}}{nqB^2} \equiv \mathbf{v}_E + \mathbf{v}_D \tag{3.58}$$

where

$$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}: \qquad \mathbf{E}\times\mathbf{B}\text{ drift}$$

$$\mathbf{v}_D = -\frac{\nabla p\times\mathbf{B}}{nqB^2} = \frac{\gamma KT}{qB}\frac{\mathbf{B}\times\nabla n}{Bn}: \qquad \text{Diamagnetic Drift}$$

Since ions and electrons drift in opposite directions, there is a current which is given by (for singly charged ions)

$$\mathbf{J}_D = en(\mathbf{v}_{Di} - \mathbf{v}_{De}) = (\gamma_i kT_i + \gamma_e kT_e)\frac{\mathbf{B}\times\nabla n}{B^2} \tag{3.59}$$

NOTES:

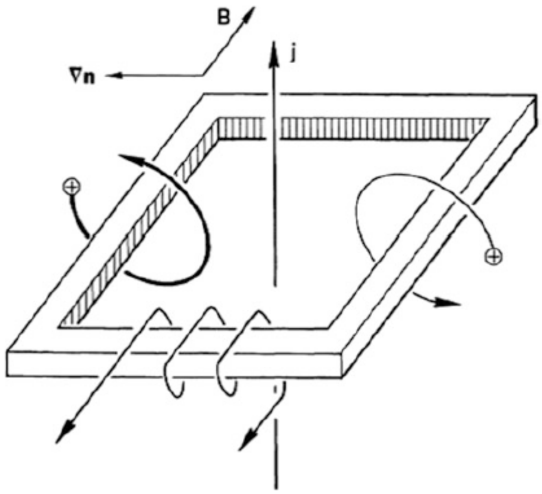

Figure 3.2:

- The current due to the diamagnetic drift produces the magnetic field which is in the opposite direction of the external field (This is why we call the drift "diamagnetic").
- The diamagnetic current does not exist in the single particle model.
  The ∇p term appears only in the fluid equation.
- The curvature drift also exists in the fluid theory.
  If the magnetic field is bent in space, the centrifugal force $\langle F_{cf} \rangle = \left\langle nmv_{\parallel}^2/R_c \right\rangle = nKT_{\parallel}/R_c$ has to be added in the fluid equation of motion.
  The magnitude of this drift velocity is $|v_R| = \frac{nKT_{\parallel}}{|q|R_c B}$.
- Since $\mathbf{v}_D$ is perpendicular to the direction of the gradient, $(\mathbf{v} \cdot \nabla)\mathbf{v} \approx 0$ is justified if $\mathbf{E} = 0$. Even if $\mathbf{E} = -\nabla\phi \neq 0$, $(\mathbf{v} \cdot \nabla)\mathbf{v}$ is still 0 if $\nabla\phi \parallel \nabla p$.
- The grad-B drift does not exist in the fluid theory. A magnetic field does not affect a Maxwellian distribution. This is because the Lorentz force is perpendicular to $\mathbf{v}$ and cannot change the energy of any particle. The most probable distribution $f(\mathbf{v})$ in the absence of $\mathbf{B}$ is also the most probable distribution in the presence of $\mathbf{B}$.

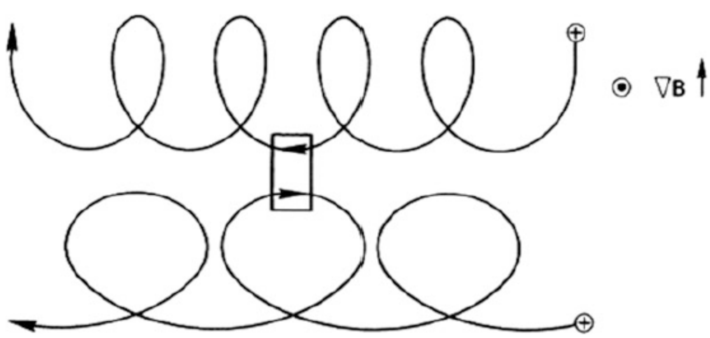

Figure 3.3:

3.4 Fluid Drifts Parallel to B

The z component of the fluid equation of motion is

$$mn\left[\frac{\partial v_z}{\partial t} + (\mathbf{v}\cdot\nabla)v_z\right] = -enE_z - \frac{\partial p}{\partial z}\,. \tag{3.60}$$

For an electron, take the limit $m \longrightarrow 0$ to get

$$0 = -\frac{\partial p}{\partial z} - enE_z\,. \tag{3.61}$$

Since $E_z = -\frac{\partial\phi}{\partial z}$,

$$0 = -\gamma KT_e\frac{\partial n}{\partial z} + en\frac{\partial\phi}{\partial z}\,. \tag{3.62}$$

Assuming electrons are isothermal ($\gamma = 1$), we have

$$\frac{1}{n}\frac{\partial n}{\partial z} = \frac{e}{KT_e}\frac{\partial\phi}{\partial z}\,. \tag{3.63}$$

An integration yields

$$\ln n = \frac{e\phi}{KT_e} + C \tag{3.64}$$

or

$$n = n_0\exp\left[\frac{e\phi}{KT_e}\right] : \quad \text{Boltzman Relation}\,. \tag{3.65}$$

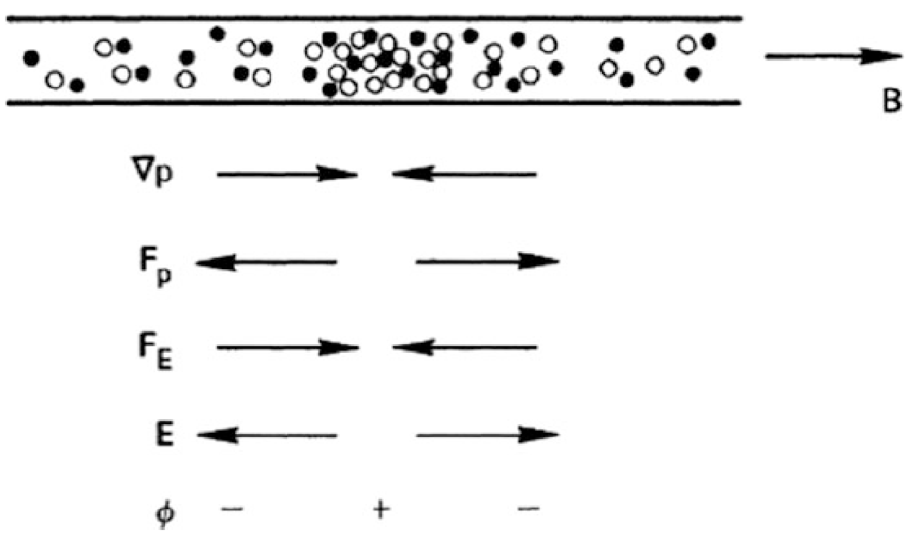

Figure 3.4: A local density clump in the plasma. Since the plasma is quasi-neutral, the gradients exist for both the electron and ion fluids. In the middle, $n > n_0$.

NOTES:

- The Boltzmann relation results from the assumption that the perturbation is very slow.

- There is balance between pressure gradient force and electrostatic force.
- $n_e = n(z)$ and $n_i \approx n_e$, but there is a finite electric field $\mathbf{E} \neq 0$.
- If $n(z) = n_0 =$ uniform, then $\phi = 0$ and $\mathbf{E} = 0$.
- The Boltzmann relation applies to each line of force separately.

3.5 The Plasma Approximation

Quasineutrality requires $n_i = n_e = n$ but $\nabla \cdot \mathbf{E} \neq 0$ (Do not use Poisson's equation to obtain $\mathbf{E}$).

- $\mathbf{E}$ is found from the equation of motion.
  $\mathbf{E}$ must adjust itself so that the orbits of the electrons and ions preserve neutrality.
  Poission's equation may be used to find the charge density.
- The plasma approximation is valid only for low frequency motion in which the electron inertia is not a factor.
- The plasma approximation is a mathematical shortcut.
- As long as motions are slow enough that both ions and electrons have time to move, it is a good approximation.