2.5 Nonuniform Electric Field

Consider static, weak, sinusoidal electric field with slow variation given by

$$\mathbf{E}(\mathbf{r}) = E(x)\hat{x} = E_0 \cos(kx)\hat{x}\,. \tag{2.79}$$

The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q\left[\mathbf{E}(x) + \mathbf{v}\times\mathbf{B}\right]\,, \tag{2.80}$$

whose transverse components are

$$\begin{cases} \dot{v}_x = +\dfrac{qB}{m}v_y + \dfrac{q}{m}E(x) \\ \dot{v}_y = -\dfrac{qB}{m}v_x \end{cases} \tag{2.81}$$

so that

$$\begin{cases} \ddot{v}_x = -\omega_c^2 v_x \pm \omega_c \dfrac{\dot{E}}{B} \\ \ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \dfrac{E}{B}\,. \end{cases} \tag{2.82}$$

Using the unperturbed orbit

$$x = x_0 + r_{\rm L}\sin\omega_c t$$

we have

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_0}{B}\cos k(x_0 + r_{\rm L}\sin\omega_c t)\,. \tag{2.83}$$

By averaging over a cycle, we obtain

$$\langle \ddot{v}_y\rangle = 0 = -\omega_c^2\langle v_y\rangle - \omega_c^2\frac{E_0}{B}\langle\cos k(x_0 + r_{\rm L}\sin\omega_c t)\rangle\ . \tag{2.84}$$

Expanding the cosine, we have

$$\cos k(x_0 + r_{\rm L}\sin\omega_c t) = \cos(kx_0)\cos(kr_{\rm L}\sin\omega_c t) - \sin(kx_0)\sin(kr_{\rm L}\sin\omega_c t)\,, \tag{2.85}$$

and for $kr_{\rm L} \ll 1$,

$$\cos k(x_0 + r_{\rm L}\sin\omega_c t) \simeq \cos(kx_0)\left[1 - \frac{1}{2}k^2 r_{\rm L}^2\sin^2\omega_c t\right] - \sin(kx_0)(kr_{\rm L}\sin\omega_c t)\,. \tag{2.86}$$

Thus we obtain

$$\langle v_y\rangle = -\frac{E_0}{B}(\cos kx_0)\left(1 - \frac{1}{4}k^2 r_{\rm L}^2\right) = -\frac{E(x)}{B}\left(1 - \frac{1}{4}k^2 r_{\rm L}^2\right)\,, \tag{2.87}$$

or

$$\mathbf{v}_{\rm E} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}\left(1 - \frac{1}{4}k^2 r_{\rm L}^2\right)\,. \tag{2.88}$$

For $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$,

$$\nabla^2\mathbf{E}(\mathbf{r}) = -k^2 E_0 e^{i\mathbf{k}\cdot\mathbf{r}} = -k^2\mathbf{E}(\mathbf{r})\,. \tag{2.89}$$

Therefore, we may generalize the result to

$$\boxed{\mathbf{v}_{\mathrm{E}} = \left(1 + \frac{1}{4} r_{\mathrm{L}}^2 \nabla^2\right) \frac{\mathbf{E} \times \mathbf{B}}{B^2}}\,. \tag{2.90}$$

NOTES:

- The second term is called the *finite Larmor radius effect*.
- $\mathbf{v}_{\mathrm{E}}$ is no longer independent of species.

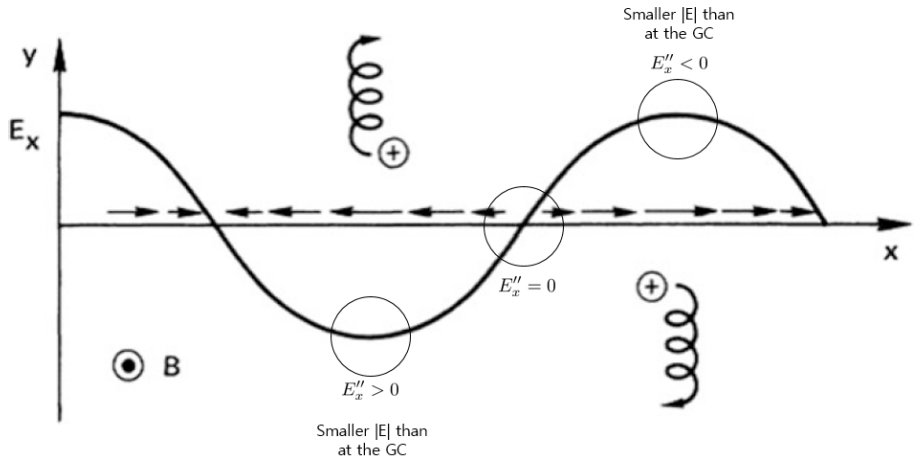

Figure 2.10:

2.6 Static Magnetic and Time–varying Electric field

Let us take $\mathbf{E}$ and $\mathbf{B}$ to be uniform in space but $\mathbf{E}$ slowly varying in time.
Let $\mathbf{E} = E\hat{x}$. The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,, \tag{2.91}$$

whose tranverse components are

$$\begin{cases} \dot{v}_x = +\dfrac{qB}{m} v_y + \dfrac{q}{m} E \\ \dot{v}_y = -\dfrac{qB}{m} v_x\,, \end{cases} \tag{2.92}$$

so

$$\begin{cases} \ddot{v}_x + \omega_c^2 v_x = \pm\omega_c \dfrac{\dot{E}}{B} \\ \ddot{v}_y + \omega_c^2 v_y = -\omega_c^2 \dfrac{E}{B}\,. \end{cases} \tag{2.93}$$

Homogeneous solutions are

$$v_{x,y} \propto e^{\pm i\omega_c t}: \quad \text{circular motion,} \tag{2.94}$$

and particular solutions are

$$\begin{cases} v_x = \pm \dfrac{\dot{E}_x}{\omega_c B} & \text{opposite for oppositely charged particle} \\ v_y = -\dfrac{E_x(t)}{B} & \mathbf{E}\times\mathbf{B} \text{ drift.} \end{cases} \tag{2.95}$$

Generally, the polarization drift (i.e., startup drift due to inertia) is given as

$$\boxed{\mathbf{v}_\mathrm{P} = \pm \frac{1}{\omega_c B}\frac{d\mathbf{E}}{dt}}, \tag{2.96}$$

and the total velocity is

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\mathrm{E} + \mathbf{v}_\mathrm{P}\,, \tag{2.97}$$

where $\mathbf{v}_\perp$ is the velocity of circular motion and the $\mathbf{v}_\mathrm{E}$ is the $\mathbf{E}\times\mathbf{B}$ drift. The polariztion drift leads to a polarization current

$$\mathbf{J}_\mathrm{P} = ne(\mathbf{v}_{ip} - \mathbf{v}_{ep}) = \frac{ne}{eB^2}(m_e + m_i)\frac{d\mathbf{E}}{dt} = \frac{\rho_m}{B^2}\frac{d\mathbf{E}}{dt}\,, \tag{2.98}$$

where ρ_m is the mass density. The total current density is

$$\mathbf{J} = \mathbf{J}_{ext} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_\mathrm{P}\,. \tag{2.99}$$

Then Ampere's Law becomes

$$\nabla\times\mathbf{H} = \mathbf{J}_{ext} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mathbf{J}_P = \mathbf{J}_{ext} + \frac{\partial}{\partial t}\left[\epsilon_0\left(1 + \frac{\rho_m}{\epsilon_0 B^2}\right)\mathbf{E}\right] = \mathbf{J}_{ext} + \frac{\partial\mathbf{D}}{\partial t} \tag{2.100}$$

where

$$\mathbf{D} = \epsilon_0\left(1 + \frac{\rho_m}{\epsilon_0 B^2}\right)\mathbf{E} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P} \tag{2.101}$$

and

$$\mathbf{P} = \frac{\rho_m}{B^2}\mathbf{E}\,. \tag{2.102}$$

NOTES:

- For a steady $\mathbf{E}$, ions and electrons move around to preserve quasineutrality.
- If $\mathbf{E}$ oscillates, an oscillating current results from the lag due to the ion inertia.

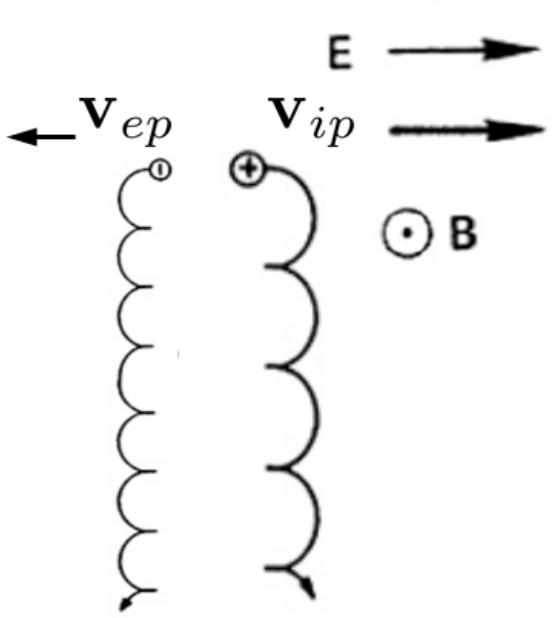

Figure 2.11:

2.7 Time-varying Magnetic Field

Let us consider the case which $\mathbf{B}$ is spatially uniform and its magnitude varies with time.

1. Since $\mathbf{B}$ varies with time, $\mathbf{E}$ will be set up and given by $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$.
2. This induced field accelerates particle and changes transverse kinetic energy.

Take the dot product of the tranverse equation of motion

$$m\frac{d\mathbf{v}_\perp}{dt} = q(\mathbf{E}_{ind} + \mathbf{v}_\perp \times \mathbf{B}) \tag{2.103}$$

with $\mathbf{v}_\perp$ to get

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\mathbf{E}_{ind}\cdot\mathbf{v}_\perp = q\mathbf{E}\cdot\frac{d\mathbf{l}}{dt}\,. \tag{2.104}$$

The change in kinetic energy during one gyration in slowly-varying $\mathbf{B}(t)$ is

$$\begin{aligned}\delta\left(\frac{1}{2}mv_\perp^2\right) &= q\int_0^{\frac{2\pi}{\omega_c}}\mathbf{E}\cdot\frac{d\mathbf{l}}{dt}dt = q\int_A \nabla\times\mathbf{E}\cdot d\mathbf{s} = -q\int_A\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{s}\\ &\simeq q\delta B\frac{\omega_c}{2\pi}\pi r_{\mathrm{L}}^2 = \frac{\frac{1}{2}mv_\perp^2}{B}\times\delta B = \mu\delta B\,.\end{aligned} \tag{2.105}$$

Since $\frac{1}{2}mv_\perp^2 = B\mu$,

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = B\delta\mu + \mu\delta B\,, \tag{2.106}$$

so that we find

$$\boxed{\delta\mu = 0} \tag{2.107}$$

which means that the magnetic moment is invariant in slowly varying magnetic fields. The magnetic moment is not a strict constant since the above treatment requires that $\dot{\mathbf{B}}$ be essentially constant throughout an orbit.
The magnetic flux is constant if μ is constant, since

$$\phi = B\pi r_{\mathrm{L}}^2 = \frac{2\pi m\mu}{q^2} = B\pi\frac{m^2 v_\perp^2}{q^2 B^2} = \frac{2\pi m}{q^2}\frac{\frac{1}{2}mv_\perp^2}{B} = \frac{2\pi m}{q^2}\mu\,. \tag{2.108}$$

The magnetic flux through a Larmor orbit is constant.

2.8 Adiabatic Invariants

2.8.1 A Brief Review of Classical Mechanics

Lagrangian

The Lagrangian function of a system is defined by

$$\boxed{L = T - V}\,, \tag{2.109}$$

where T is the kinetic energy and V is the potential energy of the system.
Lagrange's equation is given by

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0}\,, \tag{2.110}$$

where q_k is the generalized coordinate (no restriction; any quantity may define the position of the system).

Consider the case of a one-dimensional oscillator.

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{2.111}$$

$$\begin{cases} \dfrac{\partial L}{\partial \dot{x}} = m\dot{x} \\ \dfrac{\partial L}{\partial x} = -kx \end{cases}$$

The equation of motion is then

$$m\ddot{x} + kx = 0 \tag{2.112}$$

Let us find Lagrange's equation of motion for a particle moving in a plane under a central force. Choose $q_1 = r$ and $q_2 = \theta$. Then we have

$$L = T - V = \frac{1}{2}mv^2 - V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{2.113}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} = mr\dot{\theta}^2 + F_r$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0\,.$$

Therefore, the Lagrange equation yields

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 + F_r \\ \frac{d}{dt}(mr^2\dot{\theta}) &= 0\,. \end{aligned} \tag{2.114}$$

Hamiltonian

The generalized momenta is (the last equality holds only for velocity independent potential)

$$\boxed{p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}}\,. \tag{2.115}$$

For example in the above cylindrical coordinates:

$$p_2 = p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = l = \text{canonical momentum or angular momentum} \tag{2.116}$$

The Hamiltonian of a system is defined by

$$\boxed{H = \sum_k \dot{q}_k p_k - L}\,, \tag{2.117}$$

and it can be shown that

$$\boxed{H = T + V}\,. \tag{2.118}$$

Hamilton's canonical equations of motion are given as

$$\boxed{\begin{aligned} \frac{\partial H}{\partial p_k} &= \dot{q}_k \\ \frac{\partial H}{\partial q_k} &= -\dot{p}_k \end{aligned}} \tag{2.119}$$

In addition,

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} (= 0 \text{ when Hamiltonian is independent of } t) \tag{2.120}$$

For a one-dimensional harmonic oscillator,

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 \qquad V = \frac{1}{2}kx^2 \\ p &= \frac{\partial T}{\partial \dot{x}} = m\dot{x} \qquad \text{so that} \quad \dot{x} = \frac{p}{m} \\ H &= T + V = \frac{p^2}{2m} + \frac{1}{2}kx^2 \end{aligned} \tag{2.121}$$

The equations of motion

$$\frac{\partial H}{\partial p} = \dot{x} \qquad \frac{\partial H}{\partial x} = -\dot{p}$$

then read

$$\frac{p}{m} = \dot{x} \qquad kx = -\dot{p}$$

These equations may be combined into a sinle equation

$$m\ddot{x} + kx = 0$$

Action integral

In classical mechanics, whenever a system has a periodic motion, the action integral taken over a period is a constant of the motion,i.e.,

$$\boxed{\oint pdq = const.} \tag{2.122}$$

Here, p and q are the generalized momentum and coordinate.

If a slow change is made in the system, so that the motion is not quite periodic, the constant of the motion still does not change and is called an adiabatic invariant.

PROOF: (from Bellan)

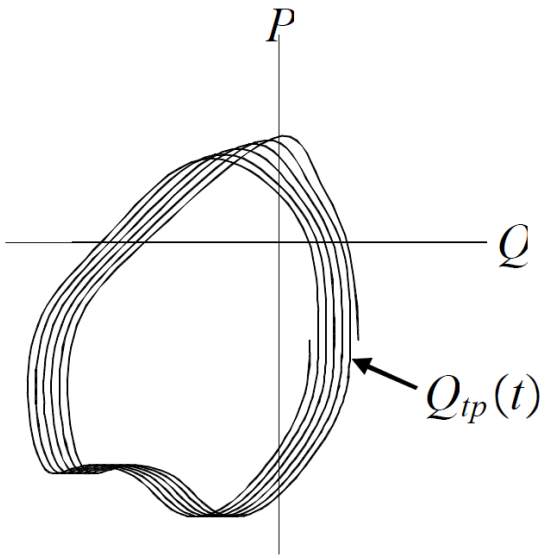

Figure 2.12:

Let the Hamiltonian depend on time via a slowly changing parameter $\lambda(t)$, so that $H = H(P, Q, \lambda(t))$. The energy is given by

$$E(t) = H(P, Q, \lambda(t)) \underbrace{\longrightarrow}_{inverted} P = P(E(t), Q, \lambda(t)) \tag{2.123}$$

It the motion is periodic, then turning point for the $(N+1)$th period will be the same as the turning point for the Nth period. When the motion is not exactly periodic, this turning point is such that $Q_{tp}(t+\tau) \neq Q_{tp}(t)$, where τ is the time interval two turning points.

$$S = \oint PdQ = \int_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} PdQ \tag{2.124}$$

$$\frac{dS}{dt} = \frac{d}{dt}\oint PdQ = \frac{d}{dt}\int_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} P(E(t), Q, \lambda(t))dQ$$

$$= \left[P\frac{dQ}{dt}\right]_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} + \int_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} \left(\frac{\partial P}{\partial t}\right)_Q dQ \tag{2.125}$$

$$= 0 + \int_{Q_{tp}(t)}^{Q_{tp}(t+\tau)} \left[\left(\frac{\partial P}{\partial E}\right)_{Q,\lambda} \frac{dE}{dt} + \left(\frac{\partial P}{\partial \lambda}\right)_{Q,E} \frac{d\lambda}{dt}\right] dQ$$

From Eq. (2.123),

$$\frac{\partial E}{\partial E} = 1 = \frac{\partial H}{\partial P}\left(\frac{\partial P}{\partial E}\right)_{Q,\lambda} \tag{2.126}$$

and

$$\frac{\partial E}{\partial \lambda} = 0 = \frac{\partial H}{\partial P}\left(\frac{\partial P}{\partial \lambda}\right)_{Q,E} + \frac{\partial H}{\partial \lambda}. \tag{2.127}$$

So

$$\frac{dS}{dt} = \oint \left(\frac{\partial H}{\partial P}\right)^{-1} \left[\frac{dE}{dt} - \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}\right] dQ. \tag{2.128}$$

From Eq. (2.123) we have

$$\frac{dE}{dt} = \underbrace{\frac{\partial H}{\partial P}}_{=\dot{Q}} \frac{dP}{dt} + \underbrace{\frac{\partial H}{\partial Q}}_{=-\dot{P}} \frac{dQ}{dt} + \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}. \tag{2.129}$$

Therefore, $dS/dt = 0$, completing the proof of adiabatic invariance.

EXAMPLE:
As an example, let us determine the adiabatic invariant for a one-dimensional oscillator.

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \tag{2.130}$$

The equation of the phase path is given by the conservation of energy $H(p,q) = E$.

$$p(q) = \sqrt{2mE - m^2\omega^2 q^2} \tag{2.131}$$

$$J = \oint p(q)dq = \text{Area of ellipse with semiaxes} \sqrt{2mE} \text{ and } \sqrt{\frac{2E}{m\omega^2}} \tag{2.132}$$

Or

$$J = \frac{2\pi E}{\omega} \tag{2.133}$$

The adiabatic invariance of J signifies that, when the parameters of the oscillator vary slowly, the energy is proportional to the frequency.

EXAMPLE:
In accelerator physics, we often have the following relation

$$J = \frac{1}{2}\left(\gamma x^2 + 2\alpha x p_x + \beta p_x^2\right) \tag{2.134}$$

or

$$1 = \frac{1}{2J}(x, p_x) \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix} = (x, p_x) A \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{2.135}$$

with $\beta\gamma = 1 + \alpha^2$. The area in the phase space defined by the above ellipse is

$$\oint p_x dx = \pi / \sqrt{\det A} = 2\pi J \tag{2.136}$$

We used the eigenvalues analysis of the matrix A.

2.8.2 The First Adiabatic Invariant, μ

The first invariant is associated with the cyclotron motion of the particle. Let

$$\begin{cases} p{=}mv_\perp r \\ \\ q{=}\theta\,. \end{cases} \tag{2.137}$$

Then the action integral is

$$\begin{aligned} \oint pdq{=}&\int_0^{2\pi} mv_\perp r_{\rm L} d\theta \\ {=}&2\pi m v_\perp r_{\rm L} \\ {=}&\frac{4\pi m}{|q|}\mu \\ {=}&\text{const.} \end{aligned} \tag{2.138}$$

2.8.3 The Second Adiabatic Invariant, J

Consider a particle trapped between two magnetic mirrors.

1. The trapped particle executes periodic motion.
2. A constant of this motion is given by $\oint m v_\parallel ds$, where ds is an element of path length along a field line.
3. However, since the guiding center drifts across field lines, the motion is not exactly periodic, and the constant of the motion becoms an adiabatic invariant.
4. This is called the longitudinal invariant J and is defined by

$$J = \int_a^b v_\parallel ds = const. \tag{2.139}$$

2.8.4 The Third Adiabatic Invariant, Φ

1. The first invariant is associated with the cyclotron motion.
2. The second invariant is associated with the longitudinal motion.
3. It would seem natural to conclude that there must be an invariant associated with the drift motion. The adiabatic invariant connected with this turns out to be the total magnetic flux inclosed by the drift surface.