2.4 Nonuniform Magnetic Field

Assume that $\mathbf{B}$ is slightly inhomogeneous so that

$$r_{\mathrm{L}} \left| \frac{\nabla \mathbf{B}}{\mathbf{B}} \right| \ll 1 \,. \tag{2.41}$$

$\mathbf{B}$ can be expressed by the Taylor series

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r} \cdot \nabla)\mathbf{B} + \cdots \,. \tag{2.42}$$

where $\mathbf{B}_0$ is the magnetic field at the guiding center and $\mathbf{r}$ is the vector from the guiding center to the instantaneous position of the particle. Then the equation of motion becomes

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m} \left[\mathbf{v} \times \mathbf{B}_0 + \mathbf{v} \times (\mathbf{r} \cdot \nabla)\mathbf{B} \right] \,, \tag{2.43}$$

where the first term on the right represents the motion in homogeneous $\mathbf{B}_0$ and the second term, which can be viewed as a correction term, arises from the perturbation of the orbit due to the inhomogeneous magnetic field. As an approximation, we use the undisturbed orbit: we replace $\mathbf{r}$ and $\mathbf{v}$ with $\mathbf{r}_0$ and $\mathbf{v}_0$ respectively, where $\mathbf{r}_0$ and $\mathbf{v}_0$ are solutions with $\mathbf{B}_0$. Then the equation of motion is

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m} \left[\mathbf{v} \times \mathbf{B}_0 + \mathbf{v}_0 \times (\mathbf{r}_0 \cdot \nabla)\mathbf{B} \right] \,, \tag{2.44}$$

- This equation can be treated by using the guiding center concept.
  The second term corresponds to an external force.

- This force is not constant because $\mathbf{r}_0$ is rotating.
  Therefore, the force must be averaged over a gyration orbit. Thus

$$\mathbf{F} = \langle q\mathbf{v}_0 \times (\mathbf{r}_0 \cdot \nabla)\mathbf{B} \rangle \,. \tag{2.45}$$

2.4.1 Grad-B Drift (Transverse gradient)

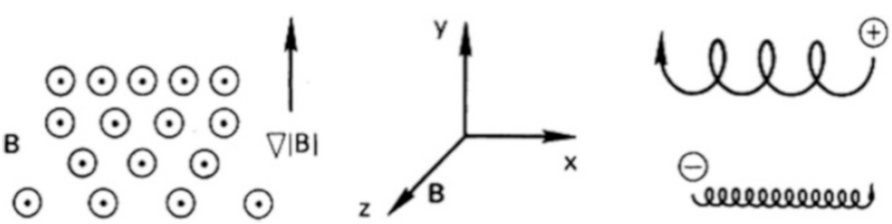

Figure 2.4:

Assume

$$\mathbf{B}(\mathbf{r}) = B(y)\hat{z} \quad (\text{make sure } \nabla \cdot \mathbf{B} = 0) \,, \tag{2.46}$$

so that we have

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r} \cdot \nabla)\mathbf{B} = \mathbf{B}_0 + y \frac{\partial \mathbf{B}}{\partial y} \,. \tag{2.47}$$

We evaluate the Lorentz force, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, compute the average force, and determine the guiding center drift.
Using undisturbed orbits,

$$\begin{cases} x = r_{\mathrm{L}} \sin \omega_c t \\ y = \pm r_{\mathrm{L}} \cos \omega_c t \,, \end{cases}$$

$$\begin{cases} v_x = v_\perp \cos \omega_c t \\ v_y = \mp v_\perp \sin \omega_c t \end{cases}$$

we can approximate the Lorentz force:

$$\begin{aligned} F_x &= q v_y B = \mp q v_\perp \sin \omega_c t \times \left(\pm r_{\mathrm{L}} \cos \omega_c t \frac{\partial B}{\partial y} \right) \\ F_y &= -q v_x B = -q v_\perp \cos \omega_c t \times \left(\pm r_{\mathrm{L}} \cos \omega_c t \frac{\partial B}{\partial y} \right) \\ F_z &= 0 \end{aligned} \quad .$$

Since

$$\begin{aligned} \langle \sin \omega t \rangle &= \frac{1}{T} \int_0^T \sin \omega t dt = 0 \\ \langle \cos \omega t \rangle &= \frac{1}{T} \int_0^T \cos \omega t dt = 0 \\ \left\langle \sin^2 \omega t \right\rangle &= \frac{1}{T} \int_0^T \sin^2 \omega t dt = \frac{1}{T} \int_0^T \frac{1 - \cos 2\omega t}{2} dt = \frac{1}{2} \\ \langle \cos^2 \omega t \rangle &= \frac{1}{T} \int_0^T \cos^2 \omega t dt = \frac{1}{T} \int_0^T \frac{1 + \cos 2\omega t}{2} dt = \frac{1}{2} \\ \langle \sin \omega t \cos \omega t \rangle &= \frac{1}{T} \int_0^T \cos \omega t \sin \omega t dt = \frac{1}{T} \int_0^T \frac{1}{2} \sin 2\omega t dt = 0 \,, \end{aligned}$$

we obtain

$$\begin{aligned} \langle F_x \rangle &= \frac{1}{T} \int_0^T F_x dt = 0 \\ \langle F_y \rangle &= \frac{1}{T} \int_0^T F_y dt \\ &= \mp q v_\perp r_{\mathrm{L}} \frac{\partial B}{\partial y} \frac{1}{2} \\ &= -\mu \frac{\partial B}{\partial y} \end{aligned}$$

where $\mu = \pm\frac{1}{2}qv_\perp r_\mathrm{L}$. Thus the drift velocity is

$$\begin{aligned}\mathbf{v}_{\nabla B} &= \frac{\mathbf{F}\times\mathbf{B}}{qB^2} \\ &= \frac{F_y\hat{y}\times B\hat{z}}{qB^2} \\ &= \frac{F_y}{qB}\hat{x} \\ &= -\frac{\mu}{qB}\frac{\partial B}{\partial y}\hat{x}\,. \end{aligned} \tag{2.48}$$

This expression may be generalized as

$$\mathbf{v}_{\nabla B} = \frac{\mu}{q}\frac{\mathbf{B}\times\nabla B}{B^2} \tag{2.49}$$

or

$$\boxed{\mathbf{v}_{\nabla B} = \frac{\frac{1}{2}mv_\perp^2}{qB}\frac{\mathbf{B}\times\nabla B}{B^2}}\,. \tag{2.50}$$

Note that this grad-B drift depends on the sign of the charge and can cause plasma currents and charge separation.

2.4.2 Curvature Drift

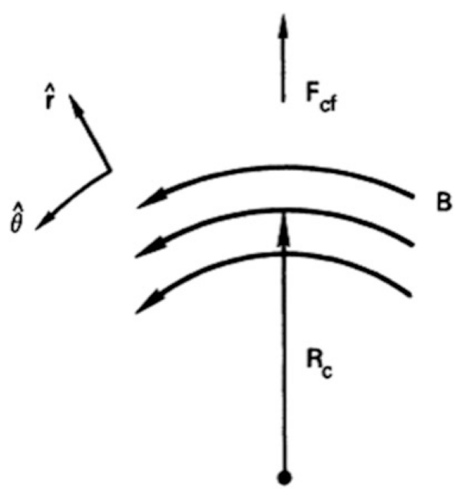

Figure 2.5:

Consider a magnetic field that has curvature.

- Such a magnetic field has a gradient perpendicular to its direction. This results in the grad-B drift.

- If the particle has a velocity along the magnetic field, it experiences a centrifugal force due to the field curvature and this force gives a drift.

The centrifugal force on a charged particle with $\mathbf{v}_\parallel$ along $\mathbf{B}$ is

$$\mathbf{F}_{cf} = m\frac{v_\parallel^2}{R_c}\hat{r} = \frac{mv_\parallel^2}{R_c}\frac{\mathbf{R}_c}{R_c}\,, \tag{2.51}$$

where $\mathbf{R}_c$ is the radius of curvature. The drift velocity due to $\mathbf{F}_{cf}$ (the curvature drift) is then

$$\boxed{\mathbf{v}_\mathrm{R} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2}}\,. \tag{2.52}$$

Form Ampere's Law, $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$, we have $\nabla\times\mathbf{B} = 0$ in a vacuum. Since For $\mathbf{B} = B(r)\hat{\theta}$,

$$\frac{1}{r}\frac{d}{dr}(rB) = 0 \tag{2.53}$$

so that

$$B(r) \propto \frac{1}{r}\,, \tag{2.54}$$

or

$$B(r) \propto \frac{1}{R_c}\,. \tag{2.55}$$

Thus

$$\frac{\nabla B}{B} = \frac{1}{B}\frac{\partial B}{\partial r}\hat{\mathbf{r}} = -\frac{\mathbf{R}_c}{R_c^2}\,. \tag{2.56}$$

Then the grad–B drift velocity may be written as

$$\mathbf{v}_{\nabla B} = \frac{\frac{1}{2}mv_\perp^2}{qB}\frac{\mathbf{B}\times\nabla B}{B^2} = \frac{\frac{1}{2}mv_\perp^2}{qB}\frac{\mathbf{B}}{B}\times\left(-\frac{\mathbf{R}_c}{R_c^2}\right) = \frac{\frac{1}{2}mv_\perp^2}{qB^2}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2}\,. \tag{2.57}$$

Hence, we have the total drift in a curved magnetic field:

$$\boxed{\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{m}{q}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)}\,. \tag{2.58}$$

Note that this drift depends on the sign of the charge and will give rise to a current and a charge separation in a plasma.

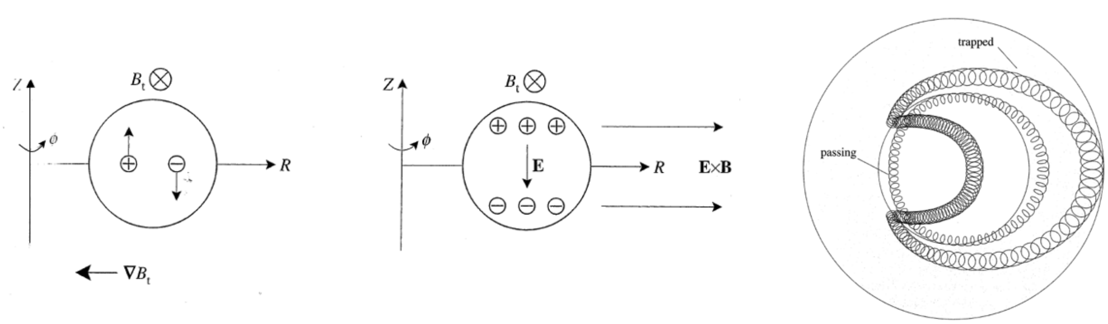

Figure 2.6:

2.4.3 Longitudinal Gradient (Magnetic Mirror)

Consider an axially symmetric magnetic field which is primarily in the z direction and whose magnitude varies in the z direction.

$$\mathbf{B} = B_r(r,z)\hat{r} + B_z(r,z)\hat{z} \tag{2.59}$$

with slow variation of B_z with z.

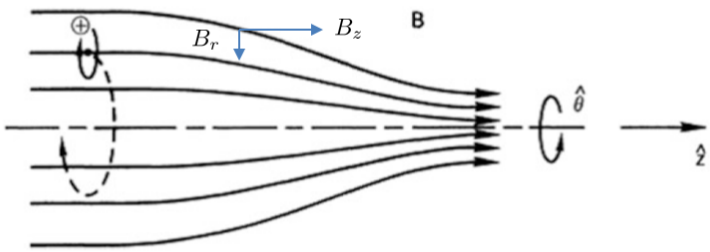

Figure 2.7:

Since

$$\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0\,, \tag{2.60}$$

$$\int_0^r \frac{\partial}{\partial r}(rB_r)dr = -\int_0^r r\frac{\partial B_z}{\partial z}dr\,. \tag{2.61}$$

If $\partial B_z/\partial z$ is given at $r=0$ and does not vary much with r, we have approximately

$$rB_r(r) = -\int_0^r r\frac{\partial B_z}{\partial z}dr \simeq -\frac{1}{2}r^2\frac{\partial B_z}{\partial z}\,, \tag{2.62}$$

or

$$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}\,. \tag{2.63}$$

The presence of this radial field gives rise to a force F_z

$$F_z = q(v_r B_\theta - v_\theta B_r) = -qv_\theta B_r = \frac{1}{2}qv_\theta r\frac{\partial B_z}{\partial z}\,, \tag{2.64}$$

and the time average is

$$\langle F_z\rangle = \mp\frac{1}{2}qv_\perp r_{\mathrm{L}}\frac{\partial B_z}{\partial z} = -\mu\frac{\partial B_z}{\partial z} \tag{2.65}$$

In general, the force on a particle along the magnetic field can be written as

$$\boxed{\mathbf{F}_\parallel = -\mu\frac{\partial B}{\partial \mathbf{s}} = -\mu\nabla_\parallel B} \tag{2.66}$$

where $d\mathbf{s}$ is the line segment along $\mathbf{B}$.
The equation of motion along $\mathbf{B}$ is

$$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial s}\,, \tag{2.67}$$

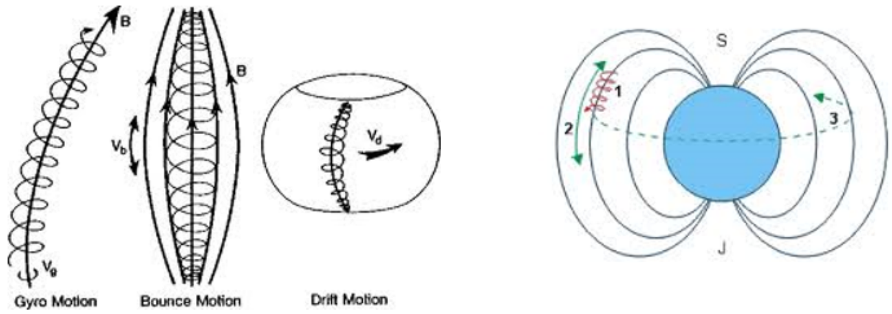

Figure 2.8:

or

$$m v_{\|} \frac{d v_{\|}}{dt} = -\mu \frac{\partial B}{\partial s} v_{\|} \,, \tag{2.68}$$

$$\frac{d}{dt}\left(\frac{1}{2} m v_{\|}^2\right) = -\mu \frac{\partial B}{\partial s}\frac{ds}{dt} \,, \tag{2.69}$$

$$\frac{d}{dt}\left(\frac{1}{2} m v_{\|}^2\right) = -\mu \frac{dB}{dt} \,. \tag{2.70}$$

Since the total kinetic energy must be conserved, we have

$$\frac{d}{dt}\left(\frac{1}{2} m v_{\|}^2 + \frac{1}{2} m v_{\perp}^2\right) = \frac{d}{dt}\left(\frac{1}{2} m v_{\|}^2 + \mu B\right) = 0 \,, \tag{2.71}$$

or

$$-\mu \frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0 \,, \tag{2.72}$$

thus

$$\boxed{\frac{d\mu}{dt} = 0} \tag{2.73}$$

which means that μ constant along the orbit, to the accuracy of this calculation. As a particle moves from one region to another one of $\mathbf{B}$, μ remains invariant while r_{L} changes. Consider a particle moving into a region of increased magnetic field. As the particle moves, it feels a force $\mathbf{F}_{\|} = -\mu \nabla_{\|} B$. Does the particle get reflected by the force, or is it lost? This will depend on its initial $v_{\|0}$ and $v_{\perp 0}$.

1. As the particle moves to the high magnetic field region $W_{\perp} = \frac{1}{2} m v_{\perp}^2$ must increase to conserve μ.

2. If $W_{\perp}$ ever reaches $W_0 = \frac{1}{2} m v_{\perp}^2$, then all the energy will be in perpendicular motion, $v_{\|}$ will vanish, and the particle will be reflected back.

A particle with $v_{\perp} = v_{\perp 0}$ and $v_{\|} = v_{\|0}$ at the midplane, where B is minimum ($B = B_{min} = B_0$), will have $v_{\perp} = v'_{\perp}$ and $v_{\|} = 0$ at its turning point. Let the field be B' at the turning point. The invariance of μ yields

$$\frac{\frac{1}{2} m v_{\perp 0}^2}{B_0} = \frac{\frac{1}{2} m v_{\perp}'^2}{B'} \tag{2.74}$$

and conservation of energy requires

$$v_0^2 = v_{\perp 0}^2 + v_{\|0}^2 = v_\perp'^2 \,. \tag{2.75}$$

Combining two equations, we find

$$\frac{v_{\perp 0}^2}{v_0^2} = \frac{v_{\perp 0}^2}{v_\perp'^2} = \frac{B_0}{B'} \,, \tag{2.76}$$

or

$$\sin^2\theta_0 = \frac{B_0}{B'} \,, \tag{2.77}$$

where θ_0 is the pitch angle at the midplane.

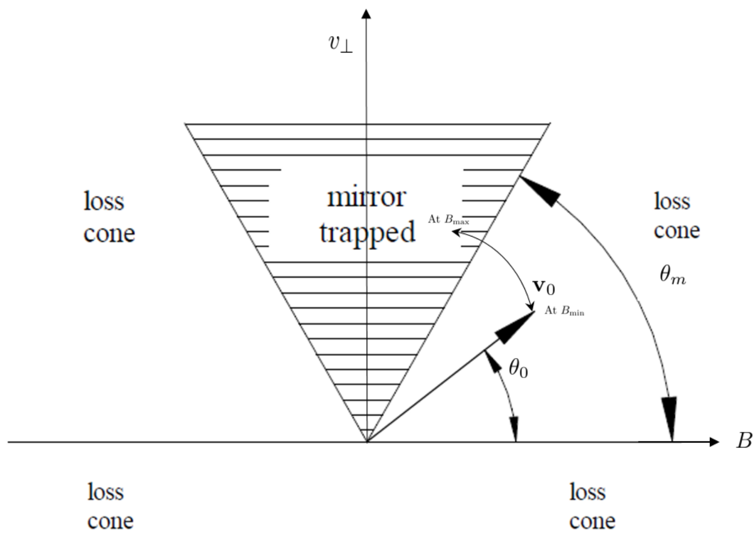

Figure 2.9: Particle with the pitch angle at the midplane θ_0 will move toward the $v_\perp$-axis as it approaches to the stronger magnetic filed. If θ_0 is less than θ_m, then even at the maximum magnetic field, there still remains the parallel velocity component.

- Particles with smaller pitch angles will be reflected in regions of higher magnetic field.
- If θ is too small, B' exceeds the maximum field B_m; and the particles are not reflected at all.
- The smallest θ of a contained particle is given by

$$\sin^2\theta_m = \frac{B_0}{B_m} \equiv \frac{1}{R} \tag{2.78}$$

  where R is the mirror ratio.
- For trapping, $\theta_0 > \theta_m$.
- This equation defines the boundary of a region in velocity space in the shape of a cone, called a loss cone. Particles with velocity vectors inside the loss cone in velocity space will escape so that a mirror–confined plasma is never isotropic.

- Mirror confinement was one of basis of major approaches to magnetic fusion. Another example of the mirror effect is the confinement of particles in the Van Allen belts.