Chapter 2

SINGLE PARTICLE MOTIONS

2.1 Introduction

$$\boxed{\text{Prescribed } \mathbf{E},\, \mathbf{B}} \longrightarrow \boxed{\text{Eq. of motion}} \longrightarrow \boxed{\mathbf{r}(t),\ \mathbf{v}(t)}$$

The "nonrelativistic" equation of motion for a particle of mass m and charge q in an electric and magnetic field is

$$\boxed{m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}\,. \tag{2.1}$$

Simple cases:

- **Field–free motion**
  For the case of zero electri and magnetic field, the velocity $\mathbf{v}_0$, the momentum $m\mathbf{v}_0$, and the kinetic energy $\frac{1}{2}mv_0^2$ are all constants of the motion.

- **Motion in a Static Electric Field**
  For the case of zero magnetic field and static electric field, the equation of motion has the solution
  $$\mathbf{v}(t) = \mathbf{v}_0 + \frac{q}{m}\mathbf{E}t\,. \tag{2.2}$$

2.2 Uniform Magnetic Field

The Lorentz equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}\,. \tag{2.3}$$

Taking the dot product of this equation with $\mathbf{v}$ yields

$$m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v} \cdot \mathbf{v} \times \mathbf{B} = 0\,, \tag{2.4}$$

so

$$\frac{1}{2}mv^2: \quad \text{const. of motion}$$

Let $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$.

From the equation of motion we have

$$m\frac{d\mathbf{v}_\parallel}{dt} + m\frac{d\mathbf{v}_\perp}{dt} = q\left[(\mathbf{v}_\parallel \times \mathbf{B}) + (\mathbf{v}_\perp \times \mathbf{B})\right] = q\mathbf{v}_\perp \times \mathbf{B} \tag{2.5}$$

which splits into two equations

$$\begin{aligned} \frac{d\mathbf{v}_\parallel}{dt} &= 0 \\ \frac{d\mathbf{v}_\perp}{dt} &= \frac{q}{m}\mathbf{v}_\perp \times \mathbf{B}\,. \end{aligned} \tag{2.6}$$

Since $v_\parallel$ is constant and $\frac{1}{2}mv^2$ is a constant of motion, we note that

$$\begin{cases} \frac{1}{2}mv_\parallel^2: & \text{const. of motion} \\ \frac{1}{2}mv_\perp^2: & \text{const. of motion} \end{cases}$$

Choose $\mathbf{B} = B\hat{z}$, then

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}_\perp \times \mathbf{B}$$

becomes

$$\begin{cases} \dot{v}_x = +\dfrac{q}{m}Bv_y \\ \dot{v}_y = -\dfrac{q}{m}Bv_x \end{cases} \tag{2.7}$$

so that we have

$$\begin{cases} \ddot{v}_x = -\left(\dfrac{qB}{m}\right)^2 v_x \\ \ddot{v}_y = -\left(\dfrac{qB}{m}\right)^2 v_y \end{cases} \tag{2.8}$$

or

$$\ddot{v}_j + \omega_c^2 v_j = 0 \tag{2.9}$$

where

$$\boxed{\omega_c = \left|\frac{qB}{m}\right|} \quad \text{cyclotron frequency}\,. \tag{2.10}$$

The solution of Eq. (2.9) is ($j = x$ or y)

$$v_j = v_\perp \exp(i\omega_c t + i\delta_j) \tag{2.11}$$

Choose (with $\pm = \mathrm{sgn}(q) = |q|/q$)

$$\begin{cases} v_x = v_\perp e^{i\omega_c t} \\ v_y = \dfrac{m}{qB}\dot{v}_x = \pm\dfrac{1}{\omega_c}\dot{v}_x = \pm i v_\perp e^{i\omega_c t} \end{cases} \tag{2.12}$$

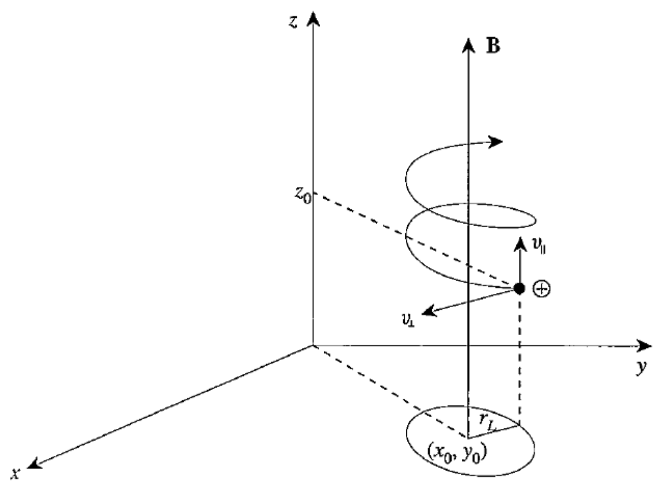

Figure 2.1:

then

$$\begin{cases} x - x_0 = -i\dfrac{v_\perp}{\omega_c} e^{i\omega_c t} \\ y - y_0 = \pm\dfrac{v_\perp}{\omega_c} e^{i\omega_c t} \end{cases}$$

Taking the real part, we obtain (x_0 and y_0 are not necessarily the initial conditions)

$$\begin{cases} x - x_0 = r_\mathrm{L} \sin \omega_c t \\ y - y_0 = \pm r_\mathrm{L} \cos \omega_c t \end{cases}$$

where

$$\boxed{r_\mathrm{L} \equiv \frac{v_\perp}{\omega_c} = \left|\frac{m v_\perp}{qB}\right|} \quad \text{Lamor radius}\,. \tag{2.13}$$

NOTES:

- **Trajectory**
  - Since $v_\parallel$ is constant, the motion in the direction parallel to the magnetic field is given by
    $$z - z_0 = v_\parallel t\,.$$
  - In the perpendicular direction, the motion is
    $$(x - x_0)^2 + (y - y_0)^2 = r_\mathrm{L}^2\,,$$
    which describes a circle in the xy–plane.
    Note different path direction for different sign of q.

- The trajectory of a particle in a uniform magnetic field is a helix with its axis parallel to the magnetic field.
  If the motion of particle is viewed by an observer moving along $\mathbf{B}$ with a velocity $v_\parallel$, the motion is a circle with its center at (x_0, y_0).
  The point $(x_0, y_0, z_0 + v_\parallel t)$ describes the locus of the center of the circle and is called a *guiding center*.

- **Pitch Angle**
  The pitch angle is defined as the angle between the velocity vector of the particle and the magnetic field:
$$\boxed{\alpha = \tan^{-1}\frac{v_\perp}{v_\parallel}}\,. \tag{2.14}$$
  Note that $v_\parallel = v\cos\alpha$ and $v_\perp = v\sin\alpha$.

- **Magnetic Moment**
  - The perpendicular motion gives rise to a circulating current
$$I = \frac{|q|\,\omega_c}{2\pi} = \frac{q^2 B}{2\pi m} \tag{2.15}$$
    The magnitude of the magnetic moment by circular motion of a charged particle is *current*$\times$*area*;
$$\mu = \frac{q^2 B}{2\pi m}\cdot \pi r_{\mathrm{L}}^2 = \frac{\frac{1}{2}mv_\perp^2}{B} = \pm\frac{1}{2} q r_L v_\perp\,. \tag{2.16}$$
  - The direction of $\boldsymbol{\mu}$ is independent of the charge and antiparallel to $\mathbf{B}$. Since the magnetic field generated by the charged particle is opposite to the externally imposed field, plasma particles tend to reduce the magnetic field (diamagnetic).
  - The magnetic moment may be regarded as a vector by using the fact that the field set up by the circulating particle opposes the ordinal field
$$\boxed{\boldsymbol{\mu} = -\frac{\frac{1}{2}mv_\perp^2}{B}\hat{b}}\,, \tag{2.17}$$
    where $\hat{b} = \mathbf{B}/B$ is an unit vector in the direction of the magnetic field.

2.3 Uniform Magnetic and Electric Fields

Consider for simplicity (without loss of generality) the magnetic and electric fields given by $\mathbf{B} = B\hat{z}$ and $\mathbf{E} = E_x\hat{x} + E_z\hat{z}$. The components of the equation of motion are

$$\begin{aligned}
\dot{v}_x &= \frac{q}{m}E_x + \frac{q}{m}Bv_y \\
\dot{v}_y &= -\frac{q}{m}Bv_x\,. \\
\dot{v}_z &= \frac{q}{m}E_z
\end{aligned} \tag{2.18}$$

From the z component equation,

$$v_z = \frac{qE_z}{m}t + v_{z0} \tag{2.19}$$

which is a straightforward acceleration along $\mathbf{B}$. The transverse components can be written as

$$\begin{aligned} \dot{v}_x &= \frac{q}{m}E_x \quad \pm \omega_c v_y \\ \dot{v}_y &= \qquad\quad \mp \omega_c v_x \,. \end{aligned} \tag{2.20}$$

Differentiating, we have

$$\begin{aligned} \ddot{v}_x &= -\omega_c^2 v_x \\ \ddot{v}_y &= \mp\omega_c \left(\frac{q}{m}E_x \pm \omega_c v_y\right) = -\omega_c^2 \left(\frac{E_x}{B} + v_y\right). \end{aligned} \tag{2.21}$$

The last equation may be written as (because $E_x/B = \text{const.}$)

$$\frac{d^2}{dt^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_c^2 \left(v_y + \frac{E_x}{B}\right) , \tag{2.22}$$

which reduces to the equation for v_y without the electric field if v_y is replaced by $v_y + E_x/B$. Therefore, we obtain

$$\begin{aligned} v_x &= v_\perp e^{i\omega_c t} \\ v_y &= \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B} \,. \end{aligned} \tag{2.23}$$

The gyrating motion is the same as before, but there is superimposed a drift velocity of the guiding center in the $-y$ direction.

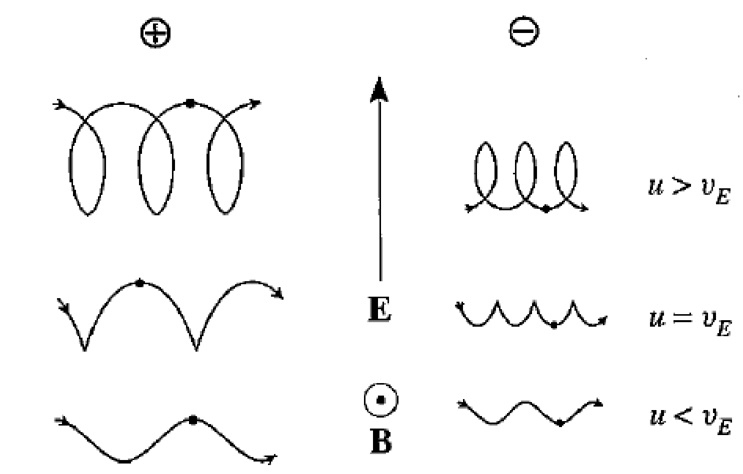

Figure 2.2: Here, $u = v_\perp$ i.e., $|\mathbf{v}_\perp|$ without electric field. $v_\perp \neq |\mathbf{v}_\perp| = |\mathbf{v}_\mathrm{E} + \mathbf{v}_\mathrm{B}(t)|$

To obtain a general formula for the drift velocity, let

$$\mathbf{B}=B\hat{z}$$

$$\mathbf{v}=\mathbf{v}_{\parallel}+\mathbf{v}_{\perp}$$

$$\mathbf{E}=\mathbf{E}_{\parallel}+\mathbf{E}_{\perp}\,.$$

The parallel component of the equation of motion is

$$\frac{d\mathbf{v}_{\parallel}}{dt}=\frac{q}{m}\mathbf{E}_{\parallel}$$

and the solution is

$$v_{\parallel}(t)=v_{\parallel 0}+\frac{q}{m}E_{\parallel}t\,,$$

which shows a motion of a constant acceleration along the magnetic field.

The perpendicular component is

$$\frac{d\mathbf{v}_{\perp}}{dt}=\frac{q}{m}(\mathbf{E}_{\perp}+\mathbf{v}_{\perp}\times\mathbf{B})\,.$$

Let $\mathbf{v}_{\perp}=\mathbf{v}_{\mathrm{E}}+\mathbf{v}_{\mathrm{B}}(t)$ where $\mathbf{v}_{\mathrm{E}}=const$, then we have

$$\frac{d}{dt}[\mathbf{v}_{\mathrm{E}}+\mathbf{v}_{B}(t)]=\frac{q}{m}(\mathbf{E}_{\perp}+\mathbf{v}_{\mathrm{E}}\times\mathbf{B}+\mathbf{v}_{\mathrm{B}}\times\mathbf{B})\,. \tag{2.24}$$

Now transform this equation to a coordinate frame moving with $\mathbf{v}_{\mathrm{E}}$ so that in this frame the particle motion is described by a purely cyclotron motion

$$\frac{d\mathbf{v}_{B}}{dt}=\frac{q}{m}\mathbf{v}_{B}\times\mathbf{B}\,. \tag{2.25}$$

This requires $\mathbf{v}_{\mathrm{E}}$ to satisfy the equation

$$q\mathbf{E}_{\perp}+q\mathbf{v}_{\mathrm{E}}\times\mathbf{B}=0\,. \tag{2.26}$$

Taking the cross product of Eq. (2.26) with $\mathbf{B}$ yields $[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})]$

$$\mathbf{E}_{\perp}\times\mathbf{B}+(\mathbf{v}_{\mathrm{E}}\times\mathbf{B})\times\mathbf{B}=0\,,$$

or

$$\mathbf{E}_{\perp}\times\mathbf{B}+(\mathbf{B}\cdot\mathbf{v}_{\mathrm{E}})\mathbf{B}-B^2\mathbf{v}_{\mathrm{E}}=0\,,$$

so (because $\mathbf{B}\perp\mathbf{v}_{\mathrm{E}}$)

$$\mathbf{v}_{\mathrm{E}}=\frac{\mathbf{E}_{\perp}\times\mathbf{B}}{B^2}\,.$$

Since $\mathbf{E}\times\mathbf{B}=\mathbf{E}_{\perp}\times\mathbf{B}$, we have

$$\boxed{\mathbf{v}_{\mathrm{E}}=\frac{\mathbf{E}\times\mathbf{B}}{B^2}}\,, \tag{2.27}$$

which is the drift velocity of the particle perpendicular to both the electric and magnetic fields.

NOTES:

- The guiding center drifts with v_{E}.
  $\mathbf{v}_{\mathrm{E}}$ is independent of q, m, and $\mathbf{v}_{\perp}$.
  No current flows or charge separation results from this motion since both positive and negative particles drift in the same direction with the same velocity.

- This result can be applied to other forces by replacing $q\mathbf{E}$ in the equation of motion by a general force $\mathbf{F}$. The guiding center drift caused by an "arbitrary constant force $\mathbf{F}$" is

$$\boxed{\mathbf{v}_{\mathrm{F}} = \frac{\mathbf{F} \times \mathbf{B}}{qB^2}}\,. \tag{2.28}$$

- For the force of gravity $m\mathbf{g}$, there is a drift

$$\mathbf{v}_{\mathrm{g}} = \frac{m}{q}\frac{\mathbf{g} \times \mathbf{B}}{B^2}\,. \tag{2.29}$$

  This drift veclocity depends on the mass and on the sign of the charge. Thus the gravitational drift motion gives rise to plasma currents and charge separation. The net current density in the plasma is

$$\mathbf{j} = nq_i\mathbf{v}_{gi} + nq_e\mathbf{v}_{ge} = n(m_i + m_e)\frac{\mathbf{g} \times \mathbf{B}}{B^2}\,. \tag{2.30}$$

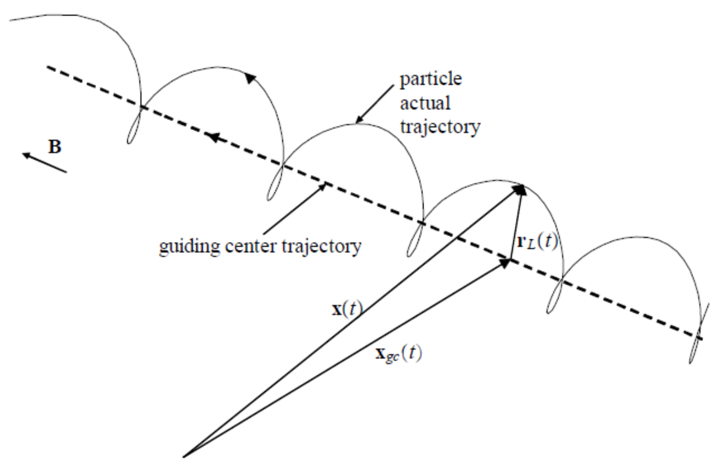

Figure 2.3: Here, $\mathbf{r} = \mathbf{r}_{\mathrm{gc}} + \boldsymbol{\rho} = \mathbf{x} = \mathbf{x}_{\mathrm{gc}} + \mathbf{r}_{\mathrm{L}}$

Appendix (Optional topic)

A guiding center is a point defined by

$$\mathbf{r}_{\mathrm{gc}} = \mathbf{r} - \boldsymbol{\rho}\,, \tag{2.31}$$

where $\mathbf{r}$ is the position vector of the particle and $\boldsymbol{\rho}$ (in some books, $\mathbf{r}_\mathrm{L}$) is the radius of curvature which is a vector from the position of the particle to the center of gyration. In the plane perpendicular to $\mathbf{B}$, we have

$$m\omega_c^2\boldsymbol{\rho} = -q\mathbf{v}\times\mathbf{B}\,. \tag{2.32}$$

Since $\omega_c^2 = q^2B^2/m^2$, this equation can be written as

$$\boldsymbol{\rho} = -\frac{\mathbf{p}\times\mathbf{B}}{qB^2}\,, \tag{2.33}$$

where $\mathbf{p}$ is the momentum of the particle. Therefore, the guiding center is given by

$$\mathbf{r}_\mathrm{gc} = \mathbf{r} + \frac{\mathbf{p}\times\mathbf{B}}{qB^2}\,. \tag{2.34}$$

Suppose that a non-magnetic force $\mathbf{F}$ is applied, then we have

$$\frac{d\mathbf{r}_\mathrm{gc}}{dt} = \frac{d\mathbf{r}}{dt} + \frac{(d\mathbf{p}/dt)\times\mathbf{B}}{qB^2}\,. \tag{2.35}$$

Using the equation of motion

$$\frac{d\mathbf{p}}{dt} = q\mathbf{v}\times\mathbf{B} + \mathbf{F}\,, \tag{2.36}$$

we obtain

$$\begin{aligned}\frac{d\mathbf{r}_\mathrm{gc}}{dt} =& \mathbf{v} + \frac{(\mathbf{F}+q\mathbf{v}\times\mathbf{B})\times\mathbf{B}}{qB^2}\\ =& \mathbf{v} + \frac{\mathbf{F}\times\mathbf{B}+q(\mathbf{v}\times\mathbf{B})\times\mathbf{B}}{qB^2}\\ =& \mathbf{v}_\parallel + \frac{\mathbf{F}\times\mathbf{B}}{qB^2}\,.\end{aligned} \tag{2.37}$$

We use

$$\begin{aligned}(\mathbf{v}\times\mathbf{B})\times\mathbf{B} &= \left\{(\mathbf{v}_\perp+\mathbf{v}_\parallel)\times\mathbf{B}\right\}\times\mathbf{B}\\ &= (\mathbf{v}_\perp\times\mathbf{B})\times\mathbf{B}\\ &= \mathbf{B}(\mathbf{B}\cdot\mathbf{v}_\perp) - \mathbf{v}_\perp B^2.\end{aligned} \tag{2.38}$$

We may write

$$\frac{d\mathbf{r}_\mathrm{gc}}{dt} = \mathbf{v}_\mathrm{gc} = \mathbf{v}_\parallel + \mathbf{v}_\mathrm{F} \tag{2.39}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp = \mathbf{v}_\parallel + \mathbf{v}_\mathrm{F} + \mathbf{v}_\mathrm{B}(t) = \mathbf{v}_\mathrm{gc} + \mathbf{v}_\mathrm{B}(t) = \text{slow} + \text{fast} \tag{2.40}$$

In this more general situation a charged particle will gyrate about $\mathbf{B}$, stream parallel to $\mathbf{B}$, have $\mathbf{E}\times\mathbf{B}$ drifts across $\mathbf{B}$, and may also have force-based drifts. The analysis is based on the assumption that all these various motions are well-separated. The assumed separation of scales is expressed by decomposing the particle motion into a fast, oscillatory component – the gyro-motion – and a slow component obtained by averaging out the gyromotion. (Bellan)