제 9 장

Nonlinear resonance and resonance overlapping

- Higher-order components in the magnetic field in a ring introduce nonlinear terms into the Hamiltonian and generate nonlinear resonances.
- This can lead to complicated motion for particles with large amplitudes of betatron oscillations.
- When the fractional part of the tune is close to $\pm 1/3$, there is a resonant structure in the phase space due to a sextupole magnet.
- For a Hamiltonian system with many resonances, they can interact with each other and lead to stochastic orbits in phase space.
- To understand this effect, we study a model called the **standard map**, that illustrates qualitative features of what can occur in a Hamiltonian system with many resonances.
- The impact on dynamics is similar, whereas the origin of the nonlinearity is diverse: nonlinear magnetic fields, RF cavities, space-charge forces among the charged particles in a bunch, or interactions between bunches.

제 1 절 The Third-Order Resonance

1.1 Hamiltonian with sextupole

The sextupole vector potential is given by

$$A_s = S(s)\left(\frac{1}{2}xy^2 - \frac{1}{6}x^3\right)$$

We will limit our analysis to one-dimensional betatron oscillations in the x direction, set $y = 0$,

$$A_s = -S(s)\frac{x^3}{6}$$

In the lowest order, the vector potential enters the Hamiltonian by

$$-\frac{eA_s}{p_0}$$

Therefore, the new Hamiltonian is

$$\mathcal{H} = \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 + \frac{1}{6}\mathcal{S}(s)x^3\,, \tag{9.1}$$

where

$$\mathcal{S} = \frac{eS}{p_0}$$

1.2 Canonical transformation

We make a transformation to the action-angle variables J_1 and ϕ_1 defined in Sect. 7.2, and then drop the subscript 1 to simplify the notation. This transformation converts the first two terms of the Hamiltonian (9.1) into a linear function of J:

$$\frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 \longrightarrow \frac{2\pi\nu}{C}J\,. \tag{9.2}$$

Transforming the last, nonlinear term we obtain:

$$\mathcal{H} = \frac{2\pi\nu}{C}J + \frac{\sqrt{2}}{3}J^{3/2}\mathcal{S}(s)\beta^{3/2}(s)\cos^3\left[\phi - \frac{2\pi\nu s}{C} + \psi(s)\right]\,. \tag{9.3}$$

Hamiltonian 구조는 변화시키지 않으면서, 다음과 같은 변수 변환을 한다.

$$\frac{dJ}{ds} = -\frac{\partial\mathcal{H}}{\partial\phi} \rightarrow \frac{dJ}{d(s\times 2\pi/C)} = -\frac{\partial(\mathcal{H}\times C/2\pi)}{\partial\phi}$$

$$\frac{d\phi}{ds} = \frac{\partial\mathcal{H}}{\partial J} \rightarrow \frac{d\phi}{d(s\times 2\pi/C)} = \frac{\partial(\mathcal{H}\times C/2\pi)}{\partial J}$$

즉, s는 θ로 변화시키고, $\mathcal{H}$에는 $C/2\pi$를 곱하여 다시 $\mathcal{H}$로 재정의 한다.

$$s\times\frac{2\pi}{C} \rightarrow \theta$$

$$\mathcal{H}\times\frac{C}{2\pi} \rightarrow \mathcal{H}$$

Therefore, θ increases by 2π every revolution. The new Hamiltonian is

$$\mathcal{H} = \nu J + V(\phi, J, \theta)\,, \tag{9.4}$$

where the perturbation V is

$$V(\phi, J, \theta) = \frac{\sqrt{2}}{3}\frac{C}{2\pi} J^{3/2} \mathcal{S}(\theta) \beta^{3/2}(\theta) \cos^3[\phi - \nu\theta + \psi(\theta)] \tag{9.5}$$
$$= \frac{C}{12\pi\sqrt{2}} J^{3/2} \mathcal{S}(\theta) \beta^{3/2}(\theta) \left\{ \cos 3\left[\phi - \nu\theta + \psi(\theta)\right] + 3\cos\left[\phi - \nu\theta + \psi(\theta)\right] \right\} .$$

The new Hamiltonian is periodic in θ with period 2π. The equations of motion for the action-angle variables are:

$$\frac{\partial J}{\partial \theta} = -\frac{\partial \mathcal{H}}{\partial \phi} = -\frac{\partial V}{\partial \phi},$$
$$\frac{\partial \phi}{\partial \theta} = \frac{\partial \mathcal{H}}{\partial J} = \nu + \frac{\partial V}{\partial J} . \tag{9.6}$$

윗식에서 θ에 대한 편미분을 사용했는데, Hamiltonian 방정식은 시간에 전미분이 사용되기 때문에, 여기서도 θ에 대한 전미분을 사용했었야 할 것 같은데...

1.3 Discussion on phase

- If we neglect the perturbation V,

$$\frac{\partial \phi}{\partial \theta} = \nu \longrightarrow \phi = \nu\theta + \phi_0$$

- After one turn

$$\theta \longrightarrow \theta + 2\pi$$
$$\nu\theta - \psi(\theta) \longrightarrow \nu \times (\theta + 2\pi) - [\psi(\theta) - 2\pi\nu] = \nu\theta - \psi(\theta)$$

  hence,

$$\phi - \nu\theta + \psi(\theta) \longrightarrow \phi + 2\pi\nu - \nu\theta + \psi(\theta)$$

  Therefore, after one turn, the argument in Eq. (9.5) changes by $2\pi\nu$.

- If the fractional part of ν is close to one-third or two-thirds, $\nu \approx n \pm 1/3$, where n is an integer, $\cos 3[\phi - \nu\theta + \psi(\theta)]$ returns to approximately to same value after θ changes by 2π.

$$3[\phi - \nu\theta + \psi(\theta)] \to 3[\phi - \nu\theta + \psi(\theta)] + 3 \times 2\pi(n \pm 1/3)$$

  The effect of this part of the perturbation accumulates with each subsequent period leading to relatively large excursions in J on the orbit.

- On the contrary,

$$[\phi - \nu\theta + \psi(\theta)] \to [\phi - \nu\theta + \psi(\theta)] + 2\pi(n \pm 1/3)$$

  so, $\cos[\phi - \nu\theta + \psi(\theta)]$ has a phase changing by $\approx \pm 2\pi/3$ after each turn, and due to continuous change of the sign of the cos function the effect of this term averages out almost to zero over many revolutions in the ring. (This term would be resonant for the tune close to an integer but, as we know, the integer values of the tune are already unstable.)

1.4 Special case of $\nu \approx n \pm 1/3$

In the rest of this section we will focus on the most interesting case when $\nu \approx n \pm 1/3$ and drop the $\cos[\phi - \nu\theta + \psi(\theta)]$ term in the perturbation,

$$V(\phi, J, \theta) = \frac{C}{12\pi\sqrt{2}} J^{3/2} \mathcal{S}(\theta) \beta^{3/2}(\theta) \cos 3[\phi - \nu\theta + \psi(\theta)] \,. \tag{9.7}$$

Let us consider a ring with one sextupole magnet of length much shorter than the ring circumference C. Without loss of generality, we can assume that the magnet is located at $\theta = 0$.

$$\mathcal{S}(\theta) = \mathcal{S}_0 \tilde{\delta}(\theta) \,, \tag{9.8}$$

where

$$\tilde{\delta}(\theta) = \sum_{k=-\infty}^{\infty} \delta(\theta + 2\pi k)$$

The requirement of the periodicity of $S(\theta)$ follows from the fact that two values of θ that differ by 2π correspond to the same position in the ring.

$$\begin{aligned} V(\phi, J, \theta) &= \frac{C}{12\pi\sqrt{2}} J^{3/2} \mathcal{S}_0 \beta^{3/2}(\theta) \tilde{\delta}(\theta) \cos 3[\phi - \nu\theta + \psi(\theta)] \\ &= \frac{1}{3} R J^{3/2} \tilde{\delta}(\theta) \cos 3[\phi - \nu\theta + \psi(\theta)] \,, \end{aligned} \tag{9.9}$$

where

$$R \equiv C\mathcal{S}_0 \beta_0^{3/2} / (4\pi\sqrt{2})$$

with $\beta_0 = \beta(0)$

$$\begin{aligned} \frac{\partial J}{\partial \theta} &= R J^{3/2} \tilde{\delta}(\theta) \sin 3[\phi - \nu\theta + \psi(\theta)], \\ \frac{\partial \phi}{\partial \theta} &= \nu + \frac{1}{2} R J^{1/2} \tilde{\delta}(\theta) \cos 3[\phi - \nu\theta + \psi(\theta)] \,. \end{aligned} \tag{9.10}$$

1.5 Evolution of J and ϕ over one turn from $\theta = -0$ to $\theta = 2\pi - 0$

We first need to integrate these equations through the delta-function kick, that is from $\theta = -0$ to $\theta = +0$.

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial \theta} &= \mathcal{J}^{3/2} \delta(\theta) \sin 3\phi \,, \\ \frac{\partial \phi}{\partial \theta} &= \frac{1}{2} \mathcal{J}^{1/2} \delta(\theta) \cos 3\phi \,, \end{aligned} \tag{9.11}$$

- Without loss of generality, we can assume that $\psi(0) = 0$.

- We rescaled the action introducing $\mathcal{J} = R^2 J$.
- We set $\theta = 0$ everywhere except in the argument of the delta function.
- We discarded the constant ν term from the second equation of (9.10) in comparison to the delta function.

1.6 How to solve?

Taking advantage of the fact that the delta function $\delta(\theta)$ is a derivative of the step function $h(\theta)$,

$$\frac{dh}{d\theta} = \delta(\theta)\,, \tag{9.12}$$

where $h(\theta)$ is equal to 1 for $\theta > 0$ and zero otherwise. By noting that

$$\frac{\partial}{\partial\theta} = \frac{dh}{d\theta}\frac{\partial}{\partial h} = \delta(\theta)\frac{\partial}{\partial h}\,, \tag{9.13}$$

we replace Eq. (9.11)

$$\begin{aligned}\frac{\partial \mathcal{J}}{\partial h} &= \mathcal{J}^{3/2}\sin 3\phi\,,\\ \frac{\partial \phi}{\partial h} &= \frac{1}{2}\mathcal{J}^{1/2}\cos 3\phi\,,\end{aligned} \tag{9.14}$$

where the independent variable h now changes from 0 to 1 when θ traverses the delta-function.

$$\boxed{\mathcal{H}(\phi,\mathcal{J}) = \frac{1}{3}\mathcal{J}^{3/2}\cos 3\phi}\,, \tag{9.15}$$

Since the Hamiltonian does not depend on the independent variable h, it is conserved, and its trajectories can be easily found from the equation

$$\mathcal{H}(\phi,\mathcal{J}) = \text{const.}$$

1.7 Map

By numerically integrating Eq. (9.14) from $h = 0$ to $h = 1$, we can generate the map that moves our particle through the sextupole magnet according to our approximate Hamiltonian.

- Action-angle variables at $\theta = -0$:

$$(\phi_1, \mathcal{J}_1)$$

- To get an entire full-turn map we then need to evaluate the equations of motion from the exit from the sextupole, $\theta = +0$, through the whole ring, to the next entrance to the magnet, $\theta = 2\pi - 0$.

$$\mathcal{J}_2 = f(\phi_1, \mathcal{J}_1), \qquad \phi_2 = g(\phi_1, \mathcal{J}_1) + 2\pi\nu \,. \tag{9.16}$$

- At the start of the n-th revolution:

$$\boxed{\mathcal{J}_n = f(\phi_{n-1}, \mathcal{J}_{n-1}), \qquad \phi_n = g(\phi_{n-1}, \mathcal{J}_{n-1}) + 2\pi\nu} \,. \tag{9.17}$$

Each pair (ϕ_n, J_n) was converted to the original canonical variables x and P_x:

$$xR/\sqrt{\beta_0} = \sqrt{2\mathcal{J}}\cos\phi, \qquad P_x R\sqrt{\beta_0} = -\sqrt{2\mathcal{J}}\sin\phi \,, \tag{9.18}$$

where for simplicity we have assumed $\alpha = 0$.

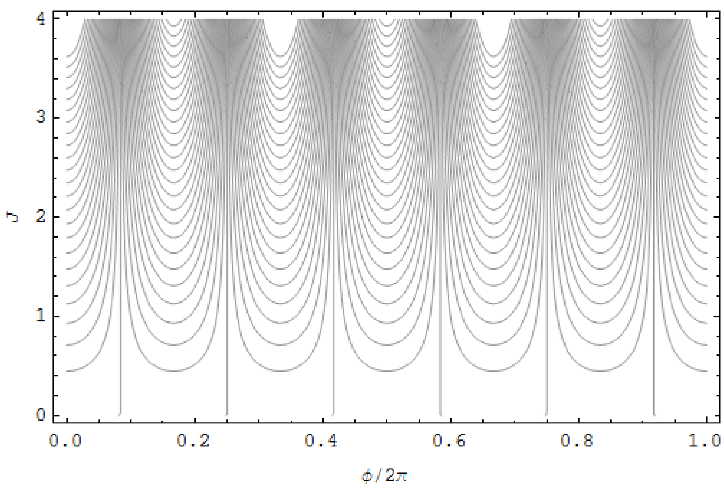

그림 9.1: $\mathcal{H}(\phi, \mathcal{J}) = \frac{1}{3}\mathcal{J}^{3/2}\cos 3\phi = $ const. 그래프를 여러개의 $\mathcal{H}$ 값에 대해 그린 것. $3\phi = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2, 9\pi/2, 11\pi/2$ 에 대해 $\mathcal{J}$ 값이 무한대로 감.

제 2 절 Standard Model and Resonance Overlapping

- As we saw in the previous section, the effect of a sextupole on betatron oscillations can be reduced to a map, Eq. (9.17), which demonstrates particle confinement near the axis and particle losses outside of the separatrix.
- Many other nonlinear beam dynamics phenomena can also be formulated in terms of Hamiltonian maps.
- In this section, we will consider one such map often called the standard, or Chirikov-Taylor, map.
- The remarkable feature of this map is that it demonstrates a transition from regular to chaotic motion in a non-integrable, time-dependent Hamiltonian system with only one degree of freedom.

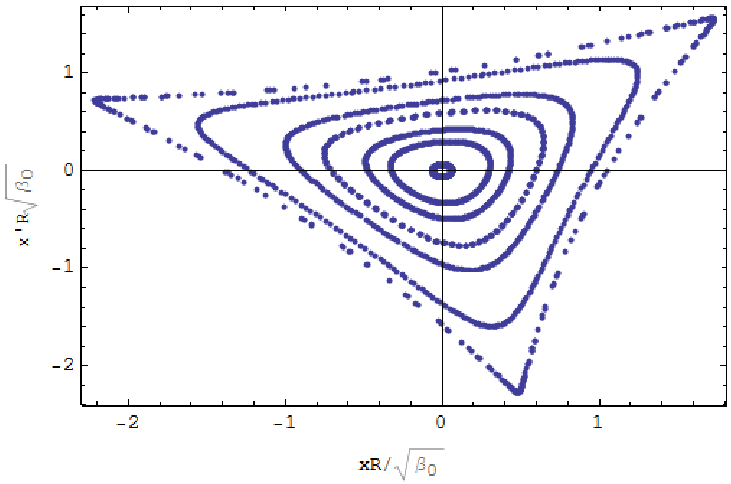

그림 9.2: 식 (9.14) 및 (9.17)을 $(\phi_1, \mathcal{J}_1) = (0, \mathcal{J}_1)$으로 놓은 7개의 초기 조건에 대해 300 번 turn 을 돌린후, 그 데이터들을 모두 모아서 실공간에 Plot 한 것.

2.1 Standard map

The standard map describes the evolution in time of a system with the following Hamiltonian:

$$H(\theta, I, t) = \frac{1}{2} I^2 + K\tilde{\delta}(t)\cos\theta\,, \tag{9.19}$$

where K is a parameter, $\tilde{\delta}(t) = \sum_{n=-\infty}^{\infty} \delta(t+n)$ is the periodic δ function that describes kicks repeating with the unit period ($T = 1$). Here, I can be considered as an action, and θ as an angle variable; they are both dimensionless. The equations of motion for I and θ are

$$\dot{I} = -\frac{\partial H}{\partial \theta} = K\tilde{\delta}(t)\sin\theta\,, \qquad \dot{\theta} = \frac{\partial H}{\partial I} = I\,. \tag{9.20}$$

$$\begin{aligned} I_{n+1} &= I_n + K\sin\theta_n\,, \\ \theta_{n+1} &= \theta_n + I_{n+1}\,. \end{aligned} \tag{9.21}$$

These equations transform the action-angle variables from their values at time $t = n$ to time $t = n + 1$. This transformation is called the standard map. [**이 map이 canonical transformation 인 것을 증명 숙제: Problem 9.1**]

2.2 Structure for only one cos term

The periodic delta-function used in Eq. (9.19) can be expanded as a Fourier series,

$$\tilde{\delta}(t) = 1 + 2\sum_{n=1}^{\infty} \cos(2\pi n t)\ . \tag{9.22}$$

$$
\begin{aligned}
H(\theta, I, t) &= \frac{1}{2}I^2 + K\cos(\theta) + 2K\cos(\theta)\sum_{n=1}^{\infty}\cos(2\pi nt) \\
&= \frac{1}{2}I^2 + K\cos(\theta) + K\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\cos(\theta - 2\pi nt)\,,
\end{aligned} \tag{9.23}
$$

where we have used the relation

$$\cos(\theta)\cos(2\pi nt) = \cos(\theta - 2\pi nt) + \cos(\theta + 2\pi nt)$$

- The two first terms on the right-hand side comprise the Hamiltonian of the pendulum.
- The sum over terms with $n \neq 0$ is a time-dependent periodic driver with the frequencies equal to $2\pi n$.

We can get some insight into the structure of the phase space by selecting only one term n:

$$\mathcal{H}(\theta, I, t) = \frac{1}{2}I^2 + K\cos(\theta - 2\pi nt)\,. \tag{9.24}$$

Making the canonical transformation: $(\theta, I) \to (\phi, J)$

$$J = I - 2\pi n\,, \qquad \phi = \theta - 2\pi nt\,, \tag{9.25}$$

we eliminate the time variables and find the new Hamiltonian [**증명 숙제: Problem 9.3**],

$$\boxed{\mathcal{H}'(\phi, J) = \frac{1}{2}J^2 + K\cos\phi}\,, \tag{9.26}$$

- Comparison between pendulum equation:

$$H(\theta, p) = \frac{p^2}{2ml^2} - \omega_0^2 ml^2\cos\theta$$

$$ml^2 \to 1, \quad -\omega_0^2 \to K, \quad \theta \to \theta + \pi$$

- Separatrix:

$$\mathcal{H}'(\phi, J) = |K|$$

  또는 아래와 같이 생각:

$$J^2 = 2(\mathcal{H}' - K\cos\phi) \geq 0 \to \text{for } \mathcal{H}' = |K|, \quad 0 \leq \phi \leq 2\pi$$

- Stable fixed points:

$$\mathcal{H}'(\phi, J) = -K$$

$$J^2 = 2(\mathcal{H}' - K\cos\phi) \geq 0 \to \text{for } \mathcal{H}' = -K, \quad \phi = \pi$$

- Maximum deviation of J on the separatrix:

$$J = \pm 2\sqrt{|K|}$$

- For the original action variable I:

$$I = 2\pi n \pm 2\sqrt{|K|}$$

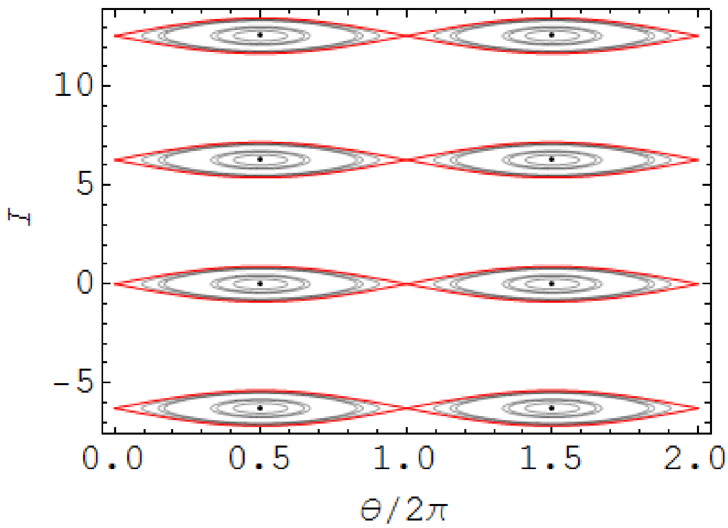

그림 9.3: 식 (9.26) 을 그린 후, I를 위아래로 $2\pi n$ 을 shift 함. 교과서와 메쓰메티카에서는 θ를 썼는데, 아마도 ϕ로 써야 맞을 것 같음. $K = 0.2$ 로 사용하여 $|K| \ll 1$ 을 만족하기 때문에, overlap 이 되지 않는다.

2.3 Overlapping of the resonance

- To understand the overall structure of the phase space of the original Hamiltonian, we can naively superimpose the phase portraits for the Hamiltonians (9.26) with different values of n.

- Such superposition makes sense only if $|K| \ll 1$, when the resonances for different values of n are well separated and, in the first approximation, do not interact with each other.

- Computer simulations of the standard map show that, indeed, as long as the distance between the islands is much larger than the width of the separatrix, then to a good approximation resonances with different values of n can be considered separately.

However, increasing $|K|$ leads to an *overlapping of the resonances* and the motion becomes much more complicated. Because the distance between the resonance is 2π and the width is $4\sqrt{|K|}$, formally overlapping occurs

$$|K| > \pi^2/4$$

But, one should not expect a drastic transition at this exact value of K. Indeed, simulations show that when $|K|$ increases, there is a gradual transformation of the flow into a regime in which the laminar orbits are destroyed and the motion becomes stochastic. Qualitatively, the transition from the *laminar* to *stochastic* motion occurs at

$$K \sim 1\,. \tag{9.27}$$

When $|K|$ becomes much larger than one, regular orbits are destroyed and the particle motion becomes chaotic. In this limit, after each kick the particle loses memory of its previous phase, and the consecutive phases θ_n can be considered to be uncorrelated.

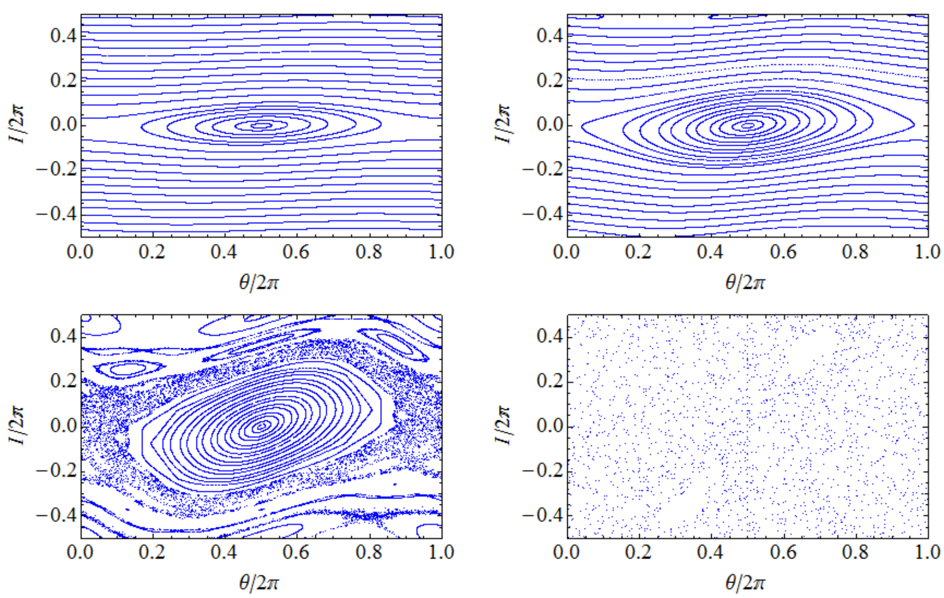

그림 9.4: 식 (9.21) 을 $K = 0.1, 0.3, 1.0, 5.0$ 로 변화시키면서 plot 함. K 값이 커지면서 점점 overlap이 되고, chaotic 해짐.

2.4 Diffusion for the action

As a result, the subsequent values of action I_n can be described as a random walk, and over many steps a statistical description of the process as a diffusion along the I axis becomes appropriate.

From Eq. (9.21), we calculate the change of the action in one step,

$$\Delta I_n = K \sin\theta_n$$

taking the square of ΔI_n and averaging it over the random phase θ_n, we obtain

$$\langle \Delta I_n^2 \rangle = \frac{1}{2} K^2 \,. \tag{9.28}$$

In a random walk, the average squares add up, and after N steps the average square of the accumulated action I_N is

$$\langle I_N^2 \rangle \approx \langle \Delta I_n^2 \rangle N \approx \frac{1}{2} K^2 N \,. \tag{9.29}$$

The linear growth of $\langle I_N^2 \rangle$ with the number of steps is a characteristic feature of the diffusion process.

제 3 절 Dynamic Aperture in Accelerators

3.1 Nonlinear fields

- We have seen in this chapter and the previous one that the severity of field errors and nonlinearities depends strongly on the *tune*.
- Nonlinear fields (both external and self-forces) have the effect of varying the tune corresponding to different particle orbits, leading to a *tune spread*.
- This means that even if a particle following the reference orbit is not near a resonance, other particles at higher amplitudes may be strongly perturbed by the nonlinear fields.
- We have not discussed *coupling* between different degrees of freedom, but that leads to more opportunities for resonances to appear (for example, if $\nu_x + \nu_y =$ integer or $\nu_x - \nu_y =$ integer).
- In a typical situation, nonlinear fields make the phase space at some distance from the reference orbit prone to stochastic motion, leading to a random walk of the particle until it is lost.

3.2 Dynamic aperture

- At best there can only be a limited region near the reference orbit where particles are properly confined. This region in phase space is called the **dynamic aperture** of the machine.
- It is computed with the help of accelerator codes by launching particles at various locations away from the reference orbit and tracking their motion.
- Rather than having a sharp boundary, the dynamic aperture is usually surrounded by an intermediate zone where the rate of diffusion which particles experience gets gradually worse.

- A related concept that focuses on the short-term tune of particles within a bunch is the analysis of **frequency maps**.
- A modern circular accelerator has many magnets that play various roles in confining the beam in the ring.
- Even as industry and researchers learn to reduce errors in the manufacture and installation of magnets, more aggressive designs and improvements in other areas tend to **make nonlinearities a major constraint** on the operation of accelerators, limiting the total charge contained in storage rings and the luminosity of colliders.