제 10 장

The kinetic equation

- In the preceding chapters we focused our attention on the motion of a single particle.
- In this chapter, we will introduce the concept of the distribution function and describe the formalism of the kinetic equation for treating large ensembles of particles in a beam.
- While this chapter focuses on deterministic Hamiltonian motion, kinetic equations in general can also include stochastic motion and damping.

제 1 절 The Distribution Function in Phase Space and the Kinetic Equation

- We begin from a simple case of one degree of freedom when each particle is characterized by two canonically conjugate variables q and p.
- The particle motion is governed by external fields, as well as interactions between the particles. In this chapter, however, we neglect the interaction effects.

1.1 Distribution function

Consider an infinitesimally small region $dq \times dp$ in the phase plane with the center located at p, q:

- Let the number of particles at time t in this phase space element be given by dN.
- This mathematically infinitesimal phase element should be considered to be physically large enough to include many particles, so that $dN \gg 1$.
- The distribution function is

$$dN(t) = f(q, p, t) dp\, dq\,. \tag{10.1}$$

We can say that the distribution function gives the density of particles in the phase space.

1.2 Time evolution of the Distribution function

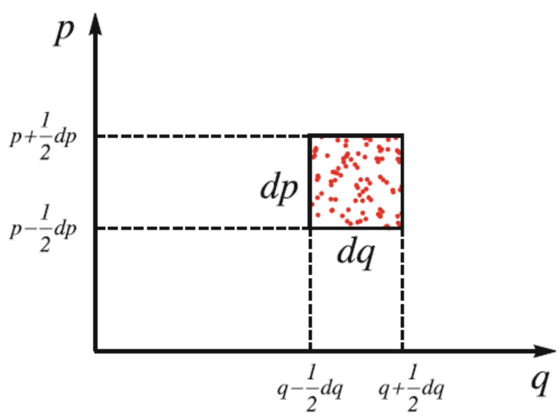

그림 10.1:

At time $t + dt$ the number of particles in the region $dq \times dp$ will change because of the flow of particles through the four boundaries of the rectangle.

1. Due to the motion in the q direction, the number of particles that flow in through the left boundary:

$$f\left(q - \frac{1}{2}dq, p, t\right) \times dp \times \dot{q}\left(q - \frac{1}{2}dq, p, t\right) dt\,. \tag{10.2}$$

   - qdt is the distance from which the flow brings new particles into the rectangle during time dt.
   - We take the values of both f and $\dot{q}$ in the middle of the left side of the rectangle

2. Similarly, the number of particles that flow out through the right boundary:

$$f\left(q + \frac{1}{2}dq, p, t\right) \times dp \times \dot{q}\left(q + \frac{1}{2}dq, p, t\right) dt\,. \tag{10.3}$$

3. Using the same logic, we calculate the number of particles which flow in through the lower horizontal boundary as

$$f\left(q, p - \frac{1}{2}dp, t\right) \times dq \times \dot{p}\left(q, p - \frac{1}{2}dp, t\right) dt\,, \tag{10.4}$$

4. The number of particles that flow out through the upper horizontal boundary:

$$f\left(q, p + \frac{1}{2}dp, t\right) \times dq \times \dot{p}\left(q, p + \frac{1}{2}dp, t\right) dt\,. \tag{10.5}$$

We are now ready to calculate the change of the number of particles in the phase volume $dq \times dp$. On one hand, this number is due to the change of the distribution function during the time interval dt,

$$dN(t+dt) - dN(t) = [f(q,p,t+dt) - f(q,p,t)]dpdq.$$

On the other hand, it is equal to the sum of the four contributions calculated above.

$$\begin{aligned}
&[f(q,p,t+dt) - f(q,p,t)]dp\,dq \\
&= f\left(q - \frac{1}{2}dq, p, t\right)\dot{q}\left(q - \frac{1}{2}dq, p, t\right)dp\,dt - f\left(q + \frac{1}{2}dq, p, t\right)\dot{q}\left(q + \frac{1}{2}dq, p, t\right)dp\,dt \\
&+ f\left(q, p - \frac{1}{2}dp, t\right)\dot{p}\left(q, p - \frac{1}{2}dp, t\right)dq\,dt - f\left(q, p + \frac{1}{2}dp, t\right)\dot{p}\left(q, p + \frac{1}{2}dp, t\right)dq\,dt\,.
\end{aligned} \tag{10.6}$$

Expanding both sides of this equation in the Taylor series and keeping only linear terms in dp, dq, dt, and then dividing both sides by dp, dq, dt, we arrive at the following result:

$$\frac{\partial f(q,p,t)}{\partial t} + \frac{\partial}{\partial q}\left[\dot{q}(q,p,t)f(q,p,t)\right] + \frac{\partial}{\partial p}\left[\dot{p}(q,p,t)f(q,p,t)\right] = 0\,. \tag{10.7}$$

This is the *continuity equation* for the function f —it is a mathematical expression of the fact that the particles in the phase space are not created and do not disappear; they are being transported from one place to another along smooth paths.

윗식을 전체 phase space 에 대해 적분하되, f 가 $q, p \to \pm\infty$에서 0 이 되다는 것을 이용하면,

$$\frac{\partial}{\partial t}\int f dpdp = \frac{\partial N}{\partial t} = \frac{dN}{dt} = 0$$

즉, 이것은 total number of particle N 이 보존된다는 것을 의미한다.

1.3 Incompressibility

- From Liouville's theorem (Sec. 3.4), the volume $dpdq$ of a space phase element does not change in Hamiltonian motion.
- Since f is proportional to the number of particles in this volume, and this number is conserved, f too is conserved, but only within a moving phase space volume element.
- The density at a given point q, p of the phase space does, however, change because a fluid element located at this point at a given time will be replaced by a new element at a later time.

From the Hamiltonian equations,

$$\frac{\partial}{\partial q}\dot{q} = \frac{\partial}{\partial q}\frac{\partial H}{\partial p}, \qquad \frac{\partial}{\partial p}\dot{p} = -\frac{\partial}{\partial p}\frac{\partial H}{\partial q}, \tag{10.8}$$

hence,

$$\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = 0$$

1.4 Vlasov equation

Combining incompressibility with Eq. (10.7)

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p} = 0\,, \tag{10.9}$$

or

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = 0 = \frac{df}{dt}$$

In accelerator and plasma physics this version of the kinetic equation is often called the *Vlasov equation*. Specifically, there are no scattering or damping terms.

In terms of Poisson bracket:

$$\frac{\partial f}{\partial t} = \{H, f\}\,. \tag{10.10}$$

For n degrees of freedom with canonical variables q_i and p_i, $i = 1, 2, \ldots, n$:

$$\frac{\partial f}{\partial t} = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i}\right)\,. \tag{10.11}$$

1.5 Normalization issue

- In some cases it is more convenient to normalize f by the number of particles N; in this case, the integral of f over the phase space is equal to one.
- With such a normalization, $f(q, p, t)dqdp$ can be understood as a probability to find a particle in the phase volume $dqdp$ in the vicinity of the phase point q, p.

제 2 절 Integration of the Vlasov Equation Along Trajectories

- We will prove by direct calculation that any distribution function which satisfies the Vlasov equation remains constant in each “fluid” element of the phase space as it moves along a particle trajectory.
- This follows from the Liouville theorem; because the phase volume of a small fluid element is conserved, the value of the distribution function equal to the ratio of the number of particles in this element to its volume does not change.

2.1 Trajectory in the extended phase space

Difference of f at two close points, at time t and $t + dt$, on this single trajectory:

$$\begin{aligned} df &= f(q + dq, p + dp, t + dt) - f(q, p, t) \\ &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p} dp\,. \end{aligned} \tag{10.12}$$

Because the two points are on the same trajectory:

$$dq = \dot{q} dt = \left(\frac{\partial H}{\partial p}\right) dt,\;\; dp = \dot{p} dt = -\left(\frac{\partial H}{\partial q}\right) dt$$

Hence,

$$df = \frac{\partial f}{\partial t} dt - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p} dt + \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} dt = 0\,, \tag{10.13}$$

or

$$\frac{df}{dt} = 0\,. \tag{10.14}$$

Here, we have used Eq. (10.9). This is the mathematical expression of the fact that f remains constant along a trajectory. This derivative, which describes changes along a trajectory, is referred to as the convective derivative, and can be written as

$$\frac{d}{dt}_{\mathrm{conv}} \equiv \frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}\,. \tag{10.15}$$

2.2 Solutions to the Vlasov equation

Knowing that f is constant along trajectories, we can find solutions to the Vlasov equation if the phase space orbits are known.

- Let $q(q_0, p_0, t)$ and $p(q_0, p_0, t)$ be solutions of the Hamiltonian equations with initial values q_0 and p_0 at $t = 0$.
- $F(q_0, p_0)$ be the initial distribution function at $t = 0$.

To find the value of f at q, p at time t we need to trace back the trajectory that passes through q, p at t, and find the initial values q_0, p_0 where it starts at $t = 0$. Hence, we need to invert the relations

$$q = q(q_0, p_0, t)\,, \qquad p = p(q_0, p_0, t)\,, \tag{10.16}$$

and find

$$q_0 = q_0(q, p, t)\,, \qquad p_0 = p_0(q, p, t)$$

The value of f at q, p at time t is then equal to the value of F at q_0, p_0:

$$f(q, p, t) = F(q_0(q, p, t), p_0(q, p, t))\,. \tag{10.17}$$

For simple trajectories, the inversion can be done analytically, and the above equation then defines f for an arbitrary initial function F.

2.3 Example: an ensemble of linear oscillators

Let us consider an ensemble of linear oscillators:

$$H(x,p) = \frac{p^2}{2} + \omega^2\frac{x^2}{2}\,. \tag{10.18}$$

The distribution function $f(x,p,t)$ for these oscillators satisfies the Vlasov equation:

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial x} - \omega^2 x\frac{\partial f}{\partial p} = 0\,. \tag{10.19}$$

Solving the Hamiltonian equations, it is easy to find the trajectory which has initial value x_0 and p_0 at $t = 0$,

$$\begin{aligned} x &= x_0\cos(\omega t) + \frac{p_0}{\omega}\sin(\omega t),\\ p &= -\omega x_0\sin(\omega t) + p_0\cos(\omega t)\,. \end{aligned} \tag{10.20}$$

Inverting these equations, we find

$$\begin{aligned} x_0 &= x\cos(\omega t) - \frac{p}{\omega}\sin(\omega t),\\ p_0 &= \omega x\sin(\omega t) + p\cos(\omega t)\,. \end{aligned} \tag{10.21}$$

If $F(x,p)$ is the initial distribution function at $t = 0$, then,

$$f(x,p,t) = F(x_0,p_0) = F\left(x\cos(\omega t) - \frac{p}{\omega}\sin(\omega t), \omega x\sin(\omega t) + p\cos(\omega t)\right)\,. \tag{10.22}$$

This solution describes a rotation of the distribution function in the phase space.

제 3 절 Action-Angle Variables in the Vlasov Equation

- The Vlasov equation has the same form independent of the choice of the canonical variables q and p.
- We will demonstrate the advantages of using the action-angle variables ϕ, J in finding a solution to the Vlasov equation.

3.1 Differential rotation

Consider a 1D system with the action-angle variables (ϕ, J) and a time-independent Hamiltonian $H(J)$. For $f = f(\phi, J, t)$,

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial J}\frac{\partial f}{\partial \phi} - \frac{\partial H}{\partial \phi}\frac{\partial f}{\partial J} = \frac{\partial f}{\partial t} + \frac{\partial H}{\partial J}\frac{\partial f}{\partial \phi} = 0\,, \tag{10.23}$$

where we have used the fact that H does not depend on ϕ. Since

$$\frac{\partial H}{\partial J} = \omega(J)$$

$$\frac{\partial f}{\partial t} + \omega(J)\frac{\partial f}{\partial \phi} = 0\,. \tag{10.24}$$

This equation is satisfied by an arbitrary function f of the following form:

$$f(\phi, J, t) = F(\phi - \omega(J)t, J)\,, \tag{10.25}$$

- This result has a simple geometrical meaning: the values of the distribution function on the orbit with a given action J rotate along this orbit with the angular frequency $\omega(J)$.
- In general, this is a differential rotation: different layers of the phase space rotate with different frequencies.

3.2 Steady-state solution

Using Eq. (10.25) we can find a general form of a steady-state distribution function that does not depend on time. Because

$$\frac{\partial f}{\partial t} = -\omega(J)\frac{\partial F}{\partial \phi}$$

$$\frac{\partial f}{\partial t} = 0 \longrightarrow F \text{ does not depend on } \phi$$

We come to the conclusion that **any function f that depends only on J is a steady-state solution to the Vlasov equation.**

3.3 Beam equilibrium in an accelerator

- The particular form of the function $f(J)$ for a beam in an accelerator cannot be found from Eq. (10.24) alone.
- In practice, the function $f(J)$ is often determined by either initial conditions (how the beam was generated or injected into an accelerator) or some slow diffusion or collision processes in the ring.
- In many cases, a negative exponential dependence of f versus J is a good approximation to the equilibrium beam state,

  $$f = \text{const}\, e^{-J/\epsilon_0} = \text{const} \exp\left(-\frac{1}{2\beta\epsilon_0}\left[x^2 + (\beta P_x + \alpha x)^2\right]\right)\,, \tag{10.26}$$

  where we have used the expression for J in a **linear magnetic lattice**. The quantity ϵ_0 is called the beam **emittance**. It is an important characteristic of the beam quality.

제 4 절 Phase Mixing

4.1 In the limit of $t \to \infty$

- From Eq. (10.25) we can draw some important conclusions about the evolution of the distribution function in the limit $t \to \infty$.

- Because ϕ is an angular variable, two values of ϕ that differ by 2π correspond to the same point in phase space. Hence F is a periodic function of ϕ with period 2π and can be expanded into the Fourier series

$$F(\phi, J) = \sum_{n=-\infty}^{\infty} F_n(J) e^{in\phi}\,, \tag{10.27}$$

where

$$F_n(J) = \frac{1}{2\pi} \int_0^{2\pi} F(\phi, J) e^{-in\phi}\, d\phi\,. \tag{10.28}$$

Since $f(\phi, J, t) = F(\phi - \omega(J)t, J)$

$$f(\phi, J, t) = \sum_{n=-\infty}^{\infty} F_n(J) e^{in[\phi - \omega(J)t]}\,. \tag{10.29}$$

- In the limit $t \to \infty$, all terms in this sum, except for $n = 0$, become rapidly oscillating functions of the action J due to the factor $e^{-in\omega(J)t}$. When calculating any integral of f over the phase space, the contribution of these terms averages out to almost zero and becomes negligible. (이 부분의 내용은 대략적으로는 이해가 가는데, 엄밀하게는 조금 더 살펴봐야 할 듯.)

- In the limit $t \to \infty$, we only need to keep the $n = 0$ term:

$$f(\phi, J, t) \longrightarrow F_0(J) \equiv \frac{1}{2\pi} \int_0^{2\pi} F(\phi, J)\, d\phi \equiv \frac{1}{2\pi} \int_0^{2\pi} f(\phi, J, 0)\, d\phi\,. \tag{10.30}$$

Here, we use $f(\phi, J, 0) = F(\phi - \omega(J)0, J) = F(\phi, J)$ This is simply the average over the angle coordinate of the initial distribution function f.

- This derivation naturally explains why the steady-state distribution function depends only on action

4.2 Phase mixing

- The mechanism that is responsible for the evolution of the distribution function to a steady state through rapid oscillations of the phase factor $e^{-in\omega(J)t}$ is called *phase mixing*.

- A linear oscillator in which ω is constant and does not depend on J is an exception: it does not exhibit phase mixing.
- Rough estimation of the time needed to approach equilibrium:
  - Let's use $\Delta\omega$ to characterize the frequency spread in the system due to the function $\omega(J)$ and the distribution of particles found in the beam.
  - The phase variation $n\omega(J)t$ at time t can be estimated as $n\Delta\omega t$.
  - The phases on different orbits start to diverge at $n\Delta\omega t \gtrsim \pi$, or $t \gtrsim \pi/n\Delta\omega$.
  - The longest time needed to mix the phases corresponds to the $n = 1$ term, giving an estimate $t \gtrsim \pi/\Delta\omega$.
  - Hence, the distribution function reaches the steady state at times $t \gg \pi/\Delta\omega$.

제 5 절 Damping and Stochastic Motion

In previous chapters we have discussed how the amplitude of motion of a single particle can decrease due to damping, or take a random walk as a result of stochastic motion. Here, we briefly describe how these effects are incorporated in the formalism of the kinetic equation.

5.1 Effect of non-conservative force

- We recalculate the convective derivative with corrections to the Hamiltonian dynamics from Eq. (3.38) due to non-conservative forces:

$$
\begin{aligned}
0 &= \frac{\partial f}{\partial t} + \sum_i \left[\frac{\partial}{\partial q_i}(\dot{q}_i f) + \frac{\partial}{\partial p_i}(\dot{p}_i f)\right] \\
&= \frac{\partial f}{\partial t} + \sum_i \left[\dot{q}_i \frac{\partial f}{\partial q_i} + \dot{p}_i \frac{\partial f}{\partial p_i} + f\frac{\partial \dot{q}_i}{\partial q_i} + f\frac{\partial \dot{p}_i}{\partial p_i}\right] \\
&= \frac{df}{dt}_{\text{conv}} + f\sum_i \frac{\partial F_i}{\partial p_i}, \qquad (10.31)
\end{aligned}
$$

  We used Eq. (3.38) with non-conservative force F_i:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + F_i$$

  thus we find that

$$\frac{df}{dt}_{\text{conv}} = -f\sum_i \frac{\partial F_i}{\partial p_i}\,. \qquad (10.32)$$

  This is consistent with the more general result Eq. (3.39) (이 부분도 잘 이해가 안감):

$$\frac{df}{dt}_{\text{conv}} = \frac{\partial f}{\partial t} + \{f, H\} + \sum_i F_i \frac{\partial f}{\partial p_i}$$

Using Eq. (3.46)

$$\frac{d}{dt}\det M = \det M(t) \sum_i \left(\frac{\partial F_i}{\partial p_i}\right)_{q(t),p(t)}$$

we can connect the evolution of the distribution function with the time derivative of the determinant of the Jacobian matrix M of the dynamic flow,

$$\frac{1}{f}\frac{df}{dt}_{\text{conv}} = -\frac{1}{\det M}\frac{d\det M}{dt}\,. \tag{10.33}$$

Integrating this equation over time, we find that $f(t)$ evaluated along a particle trajectory scales in time as the inverse of the determinant of the matrix $M(t)$:

$$\frac{f(q_i,p_i,t)}{f(q_i,p_i,0)} = \frac{1}{[\det M(t)]} \tag{10.34}$$

where $\det M(0) = 1$. This is consistent with the notion that the phase space density increases only when trajectories converge in phase space due to non-Hamiltonian dynamics.

- For one degree of freedom

$$F = -\gamma\dot{x} = -\gamma p$$

$$\frac{df}{dt} = -f\frac{\partial F}{\partial p} = -f(-\gamma) = \gamma f \tag{10.35}$$

Hence

$$f(x(t),p(t),t) = f(x(0),p(0),0)\,e^{\gamma t}\,. \tag{10.36}$$

5.2 Random kicks

- Random kicks with a small correlation time can also be incorporated into the formalism of the distribution function in a natural way if the coordinates are chosen so that only the momenta are directly impacted by the kicks.

- Because these kicks lead to a random walk of individual particles, this appears in the distribution function as a diffusion process when a large ensemble is considered

- We consider a single degree of freedom and very short time scales so that the dynamics have

  1. a negligible impact
  2. uncorrelated random kicks with typical magnitude Δp
  3. a typical time Δt between kicks lead to a random walk with

  $$\left\langle [p(t)-p_0]^2 \right\rangle = (t-t_0)\left\langle (\Delta p)^2/\Delta t \right\rangle$$

  where $p(t_0) = p_0$.

- This process, convolved with the initial distribution, leads to a spreading out of the distribution function.

- Statistically, it can also be described as the result of a differential operator

$$\frac{\partial f}{\partial t} = D_{\mathrm{s}} \frac{\partial^2 f}{\partial p^2}\,, \tag{10.37}$$

  where

$$D_s = \left\langle (\Delta p)^2 / \Delta t \right\rangle \tag{10.38}$$

5.3 Vlasov-Fokker-Planck equation

- Because the above expression of the dynamics depends on infinitesimal time scales, it is only necessary to add back the full dynamics by replacing $\partial f / \partial t$ with the convective derivative (이 말도 명확하게 이해는 가지 않음.)—for completeness we include the correction from frictional forces:

$$\left.\frac{df}{dt}\right._{\mathrm{conv}} = -f \frac{\partial F}{\partial p} + D_{\mathrm{s}} \frac{\partial^2 f}{\partial p^2}\,. \tag{10.39}$$

  The impact of this differential operator will be mixed with that of the particle dynamics to yield a spread in both momentum and position, especially when the frequency of motion is fast compared to the impact of the scattering.

- For a one-dimensional system and the simple form of damping $F = -\gamma p$,

$$\dot{p} = -\frac{\partial H}{\partial q} + F = -\frac{\partial H}{\partial q} - \gamma p$$

  we can expand this to find the partial time derivative:

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{\partial H}{\partial p}\frac{\partial f}{\partial x} + \frac{\partial H}{\partial q}\frac{\partial f}{\partial p} + \gamma p \frac{\partial f}{\partial p} + \gamma f + D_{\mathrm{s}} \frac{\partial^2 f}{\partial p^2} \\ &= -\{f, H\} + \frac{\partial}{\partial p}(\gamma p f) + D_{\mathrm{s}} \frac{\partial^2 f}{\partial p^2}\,. \end{aligned} \tag{10.40}$$

  The second term in the final expression (related to damping) combines the effect of the distribution function having a convective derivative (as found above) with the fact that the flow in phase space is itself no longer fully defined by the Poisson bracket, as seen in Eq. (3.39).