제 8 장

Magnetic field and energy errors

- Due to small magnetic field errors, the magnetic field in any real machine differ from the ideal linear lattice.
- Also, the particle energy in the beam deviates from the reference energy for which the ideal closed orbit is designed.
- In this chapter, we study how the *dipole and quadrupole field errors* as well as *small energy deviations* affect the orbits.

제 1 절 Closed Orbit Distortions

- Assumptions:
  - Design vertical magnetic field: $B_0(s)$
  - Magnetic field with error in $\hat{\boldsymbol{y}}$: $\mathbf{B}(s) = \hat{\boldsymbol{y}}\,[B_0(s) + \Delta B(s)]$
  - Deviation from the ideal field is small: $|\Delta B| \ll B_0$
  - Our curvilinear coordinate system (x, y, s) is associated with the ideal reference orbit of B_0, which is the line $x = y = 0$
  - The vector potential ΔA_s of the error field:

$$A_s = -B(s)x\left(1 - \frac{x}{2\rho}\right) \to \Delta A_s = -\Delta B(s)x \text{ to the first order}$$

- From the Hamiltonian (6.10):

$$\mathcal{H}_x = \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 + \frac{e\Delta B(s)}{p_0}x\,, \tag{8.1}$$

where,

$$K = \frac{1}{\rho^2} + \frac{eG}{p_0}$$

The differential equation for x:

$$x'' + K(s)x = -\frac{e\Delta B(s)}{p_0}\,. \tag{8.2}$$

– This equation describes general betatron oscillations, but can also be used to find a new closed orbit in the perturbed magnetic field.

– Because the perturbation is small, this new orbit should be close to the old one; it is obtained as a periodic solution to Eq. (8.2).

– We denote this solution by $x_0(s)$; it is called the *closed orbit distortion.*

– By definition, closed orbits must satisfy the periodicity condition $x_0(s+C) = x_0(s)$.

- Calculation of $x_0(s)$:
We first consider the case of a field perturbation localized at one point,

$$\Delta B(s) = \Delta\mathcal{B}_0\delta(s-s')$$

Since the right-hand side of Eq. (8.2) (perturbation term) is equal to zero everywhere except for the point $s = s'$, we seek a solution in the following form:

$$x_0(s) = A\sqrt{\beta(s)}\cos[\psi(s) - \psi_0]\,. \tag{8.3}$$

- Two requirements:

1. x_0 should be continuous at $s = s'$:
We choose

$$\psi_0 = \psi(s') + \pi\nu \text{ at } s = s'$$

$$\psi(s) = \psi(s') \to \psi(s') + 2\pi\nu \text{ when } s = s' \to s' + C$$

Then when $s = s'$

$$\psi(s) = \psi(s')$$

$$x_0(s') = A\sqrt{\beta(s')}\cos[-\pi\nu]$$

After one turn around the ring,

$$x_0(s'+C) = A\sqrt{\beta(s'+C)}\cos[\pi\nu] = x_0(s')$$

Hence,

$$x_0(s) = A\sqrt{\beta(s)}\cos[\psi(s) - \psi(s') - \pi\nu]\,. \tag{8.4}$$

2. A jump in the first derivative of x_0 at s':

$$\int_{s'-\epsilon}^{s'+\epsilon} [x'' + K(s)x]\,ds = -\int_{s'-\epsilon}^{s'+\epsilon} \frac{e\Delta B(s)}{p_0} ds$$

$$x'(s'+\epsilon) - x'(s'-\epsilon) + K(s')[\cancel{x(s'+\epsilon) - x(s'+\epsilon)}] = -\frac{e\Delta\mathcal{B}(s')}{p_0}$$

Since $s' \le s \le s' + C$,

$$x'(s'+\epsilon) \to x'(s'), \text{ and } x'(s'-\epsilon) \to x'(s'+C)$$

we have

$$x_0'(s') - x_0'(s'+C) = -\frac{e\Delta\mathcal{B}_0(s')}{p_0}\,. \tag{8.5}$$

Substituting Eq. (8.4) into Eq. (8.5) we find

$$A = -\frac{\sqrt{\beta(s')}}{2\sin(\pi\nu)}\frac{e\Delta\mathcal{B}_0(s')}{p_0}\,. \tag{8.6}$$

- Extending the delta-function perturbation to the case of an arbitrary $\Delta B(s)$: We represent $B(s)$ as a superposition of delta-function contributions using

$$\Delta B(s) = \int_s^{s+C} ds'\Delta B(s')\delta(s-s')\,. \tag{8.7}$$

  Because the solution of the following equation,

$$x'' + K(s)x = -\frac{e\Delta B(s)}{p_0} = -\frac{e}{p_0}\int_s^{s+C} ds'\Delta B(s')\delta(s-s')$$

  x_0 is linear in ΔB, we need to add contributions from all locations s', integrating the right-hand side of Eq. (8.4) over s':

$$x_0(s) = \frac{-e}{2p_0\sin(\pi\nu)}\int_s^{s+C} ds'\Delta B(s')\sqrt{\beta(s)\beta(s')}\cos[\psi(s)-\psi(s')-\pi\nu]\,. \tag{8.8}$$

- Integer resonance: An important immediate consequence of this formula is that integer values for the tune ν are not allowed in a realistic magnetic lattice. Such a lattice would be unstable with respect to small errors of the ideal magnetic field.

- General solution:

  - We found the x_0 (closed orbit distortion, 만약 아무런 이중극자 에러가 없다면 $x_0 = 0$) as a particular solution of the inhomogeneous differential equation.
  - The general solution is

$$x(s) = x_0(s) + \xi(s)\,. \tag{8.9}$$

  - $\xi(s)$ satisfies the homogeneous equation

$$\xi'' + K(s)\xi = 0\,, \tag{8.10}$$

    which is the equation for the horizontal betatron oscillations. The meaning of $\xi(s)$ is the betatron oscillation around the perturbed closed orbit $x_0(s)$.

- Action-angle variables for the new Hamiltonian:
  - The unperturbed coordinate x and the momentum $P_x = x'$ are measured relative to the ideal closed orbit $x = 0$.
  - With the perturbed magnetic field, both the distance and the angle has to be measured relative to the perturbed closed orbit $x0(s)$.
  - The new action is obtained by simply replacing $x \to x - x_0(s)$ and $P_x \to P_x - x_0'(s)$:

$$J(x, P_x, s) = \frac{1}{2\beta}\left\{(x - x_0)^2 + [\beta(P_x - x_0') + \alpha(x - x_0)]^2\right\}. \tag{8.11}$$

- Notes:
  - As the perturbation of the fields becomes larger, higher-order terms that were neglected in this derivation become more important.
  - At some point, it may even become impossible to find a closed orbit at all.
  - The case of the horizontal perturbation, $B(s) = \hat{\boldsymbol{y}}B_0(s) + \hat{\boldsymbol{x}}\Delta B(s)$, can be considered in a similar fashion: to lowest order the perturbation propagates into the Hamiltonian $\mathcal{H}_y$, and the perturbed closed orbit gets a distortion $y_0(s)$ in the vertical direction.

제 2 절 Effect of Energy Deviation

- Another effect that causes a distortion of the closed orbit is the deviation of the particle energy from the nominal one

- We can directly adapt the results of the previous section to find the distortion for a particle with a relative energy that differs from the nominal one by $\eta = (p - p_0)/p_0$.

- From the Hamiltonian (6.10), we see that it has a term $-\eta x/\rho$ that couples η to the horizontal (x) motion.

$$\mathcal{H}_x = \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 - \frac{\eta}{\rho}x\,. \tag{8.12}$$

- Comparing with Eq. (8.1)

$$\Delta B \longrightarrow -\frac{\eta p_0}{e\rho}\,. \tag{8.13}$$

  and we find the orbit distortion caused by the energy deviation η,

$$x_0(s) = D(s)\eta\,, \tag{8.14}$$

where the function D is

$$D(s) = \frac{1}{2\sin(\pi\nu)} \int_s^{s+C} ds' \frac{\sqrt{\beta(s)\beta(s')}}{\rho(s')} \cos[\psi(s) - \psi(s') - \pi\nu] \, . \tag{8.15}$$

This function is called the *dispersion function* of the ring, and it too is a periodic function of s.

- Using Eqs. (8.11) and (8.14), we can also find the action variable for a particle with an energy deviation η,

$$J(x, P_x, \eta, s) = \frac{1}{2\beta} \left([x - \eta D(s)]^2 + \{\beta[P_x - \eta D'(s)] + \alpha[x - \eta D(s)]\}^2 \right) \, . \tag{8.16}$$

제 3 절 Quadruple Errors

3.1 Perturbed Hamiltonian

- Let us assume that due to a perturbation of the quadrupole field the ideal focusing strength $K(s)$ is changed to $K(s) + \Delta K(s)$, where $|\Delta K| \ll |K|$.

- The perturbed Hamiltonian is

$$\mathcal{H} = \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 + \frac{1}{2}\epsilon\Delta K(s)x^2 \,, \tag{8.17}$$

  where we have introduced a formal smallness parameter ϵ, which will be set to unity at the end of the calculation. (계산 과정에서 매우 작은 항이라는 것을 표시하기위해서 도입한 것이고, 계산의 마지막 단계에서는 1로 놓으면 됨.)

- With the proper choice of function $K(s)$, this Hamiltonian can be applied to both horizontal (x) and vertical (y) coordinates.

- Since we know that the focusing function $K(s)$ determines the betatron oscillations in the system, it is clear that changing the focusing strength would result in the perturbation of the beta function and, hence, the tune of the ring.

3.2 Action-angle variables

- Transformation to the action-angle variables J and ϕ casts the first two terms of the Hamiltonian into J/β:

$$\frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 \longrightarrow \frac{J}{\beta(s)} \,. \tag{8.18}$$

- For the perturbed term in the Hamiltonian, we express x in terms of J and ϕ using

$$x = \sqrt{2\beta J}\cos\phi$$

$$\begin{aligned}\mathcal{H} &= \frac{J}{\beta(s)} + \epsilon\Delta K(s)J\beta(s)\cos^2\phi \\ &= J\left(\frac{1}{\beta(s)} + \frac{1}{2}\epsilon\Delta K(s)\beta(s)\right) + \frac{1}{2}\epsilon\Delta K(s)J\beta(s)\cos 2\phi \,,\end{aligned} \tag{8.19}$$

  where we have split the perturbation term into an averaged part and one oscillating using

$$\cos^2\phi = \frac{1}{2}(1 + \cos 2\phi)$$

- The last term in the above equation is denoted by $\epsilon V(\phi, J, s)$:

$$V(\phi, J, s) = \frac{1}{2}\Delta K(s)J\beta(s)\cos 2\phi \,. \tag{8.20}$$

3.3 Perturbation theory based on canonical transformations

Here, the goal is to eliminate the perturbation term from the transformed Hamiltonian. There is no general method that completely eliminates the angular dependence on ϕ. However, this dependence can be made of higher order in ϵ.

- Step 1: A canonical transformation to new variables, $(\phi, J) \to (\xi, I)$ using the generating function of the second type.

$$F_2(\phi, I, s) = \phi I + \epsilon G(\phi, I, s)\,, \tag{8.21}$$

  We will determine the appropriate function G.

- We obtain the following relations between the old and new variables:

$$\xi = \phi + \epsilon G_I(\phi, I, s)\,, \qquad J = I + \epsilon G_\phi(\phi, I, s)\,, \tag{8.22}$$

  where

$$G_I = \frac{\partial G}{\partial I}, \quad G_\phi = \frac{\partial G}{\partial \phi}$$

- Because G is multiplied by the small parameter ϵ, the difference between the old and the new variables is small. We can solve these equations to the first order in ϵ: 즉, 윗식에서 함수 G 안에 있는 I 를 J 로 근사적으로 대체.

$$\xi \approx \phi + \epsilon G_I(\phi, J, s)\,, \qquad I \approx J - \epsilon G_\phi(\phi, J, s)\,, \tag{8.23}$$

- New Hamiltonian:

$$H' = H + \frac{\partial F_2}{\partial t} \to \mathcal{H}_1 = \mathcal{H} + \frac{\partial G}{\partial s}$$

$$\mathcal{H}_1 = I\left(\frac{1}{\beta} + \frac{1}{2}\epsilon\Delta K\beta\right) + \epsilon V + \frac{1}{\beta}\epsilon G_\phi + \epsilon G_s + O(\epsilon^2)\,. \tag{8.24}$$

- We can cancel out the angle-dependent part of the perturbation V in the new Hamiltonian by choosing G in such a way that the ϕ-dependent terms in Eq. (8.24), which are all linear in ϵ, cancel:

$$V + \frac{1}{\beta}G_\phi + G_s = 0\,. \tag{8.25}$$

- Note that because the old and new variables differ by small terms of the order of ϵ, we can write V as if it were a function of the variables (ϕ, I, s) to match the form of the function G from the generating function. 즉, 새 변수와 기존 변수 사이의 차이는 ϵ 수준 정도 밖에 안나기 때문에, ϵ 의 1차항 수준에서는 구분이 무의미하다는 의미.

- Solution:

$$G(\phi, I, s) = -\frac{I}{4\sin(2\pi\nu)}\int_s^{s+C} ds'\Delta K(s')\beta(s')\sin 2[\phi - \psi(s) + \psi(s') - \pi\nu]\,. \tag{8.26}$$

  윗식을 직접 식 (8.25)에 대입하여 증명할 수 있음 (**숙제**).

– The function G should be periodic in s with the period equal to the ring conference C.

– 실제로, 윗 식의 적분에서 $s' \to s' + C$ 및 $s \to s + C$ 로 각각 치환을 해도 결과가 변하지 않는다.

– To avoid a singularity in the formula we need to require that V is not equal to integer or half-integer values.

• New Hamiltonian with the solution G:

$$\mathcal{H}_1 = I \left(\frac{1}{\beta} + \frac{1}{2} \epsilon \Delta K \beta \right) , \tag{8.27}$$

– We have neglected higher order terms.

– Because $\mathcal{H}_1$ does not depend on the angle ξ, the action I is an integral of motion.

– New action is given by Eq. (8.23)

$$I = J + \frac{\epsilon J}{2 \sin(2\pi\nu)} \int_s^{s+C} ds' \Delta K(s') \beta(s') \cos 2[\phi - \psi(s) + \psi(s') - \pi\nu] . \tag{8.28}$$

3.4 New beta function and betatron phase

• New modified beta function β_1 and corresponding betatron phase ϕ_1:

– 단순하게, $\xi \to \phi_1$ 으로 생각하기 쉬운데...그렇지 않음.

– 또는, $\mathcal{H}_1 \to I/\beta_1$ 으로 생각하기 쉬운데...이 역시 반드시 그렇지 않음.

– 7장에 나온 예를 볼것. 식 (7.14) 에서 식 (7.19)의 내용.

• The proper definition of the beta function and betatron phase comes from Eqs. (7.8)–(7.10). These expressions should hold equally in the new action-angle coordinates by taking $J \to I$, $\beta \to \beta_1$ and $\phi \to \phi_1$:

$$\begin{aligned} x &= \sqrt{2\beta_1 I} \cos\phi_1 , \\ P_x &= -\sqrt{\frac{2I}{\beta_1}} \left(\sin\phi_1 + \alpha_1 \cos\phi_1 \right) , \\ I &= \frac{1}{2\beta_1} \left[x^2 + (\beta_1 P_x + \alpha_1 x)^2 \right] , \end{aligned} \tag{8.29}$$

where we again define $\alpha_1 = -\beta_1'/2$. 즉, quadrupole 에러가 있다고 하더라도, 기본적으로는 Hill's equation 을 만족하면서 betatron oscillation 을 함. 물론, 입자의 운동 자체는 주기적이지 않음.

- Even though both β_1 and ϕ_1 are unknown, we can overcome this by using the fact that they are only functions of s, and that the above expressions have to hold true for all combinations of x and P_x. Thus we may limit our attention to the case $\phi = \pi/2$. This requires $x = 0$, so ϕ_1 is constrained to be $\pi/2$ as well and

$$J = \frac{1}{2}\beta P_x^2 \,, \qquad I = \frac{1}{2}\beta_1 P_x^2 \,. \tag{8.30}$$

which implies

$$\beta_1 = \frac{\beta}{J} I|_{\phi=\pi/2} \,. \tag{8.31}$$

즉, β는 s만의 함수로 가정하면, 그 값은 임의이 ϕ 를 잡아도 변하지 안을 것임.

- We write Eq. (8.28) as $I = J + \epsilon\Delta J$, and write the new beta function to first order in ϵ as

$$\beta_1 = \beta + \epsilon\Delta\beta$$

Since

$$\beta + \epsilon\Delta\beta = \frac{\beta}{J}\left(J + \epsilon\Delta J\right) \text{ at } \phi = \pi/2 \tag{8.32}$$

Therefore,

$$\begin{aligned}
\Delta\beta(s) &= \frac{\beta(s)}{J}\Delta J|_{\phi=\pi/2} \\
&= \frac{\beta(s)}{2\sin(2\pi\nu)} \int_s^{s+C} ds' \Delta K(s')\beta(s') \cos 2[\pi/2 - \psi(s) + \psi(s') - \pi\nu] \\
&= -\frac{\beta(s)}{2\sin(2\pi\nu)} \int_s^{s+C} ds' \Delta K(s')\beta(s') \cos 2[-\psi(s) + \psi(s') - \pi\nu] \,,
\end{aligned} \tag{8.33}$$

where

$$\cos(\pi + \theta) = -\cos\theta$$

- Based on the above equation, one should avoid integer or half-integer values of the tune — they are unstable with respect to errors in the focusing strength of the lattice (**half-integer resonance**).

- Having found the correction to the beta function, we can find the correction to the tune by using

$$\nu = \frac{1}{2\pi} \int_0^C \frac{ds}{\beta(s)} \tag{8.34}$$

제 4 절 몇 가지 다른 교과서의 유용한 그림

- S. Y. Lee 책의 그림: S. Y. Lee 책은 우리 교과서의 x를 y로 표기함.

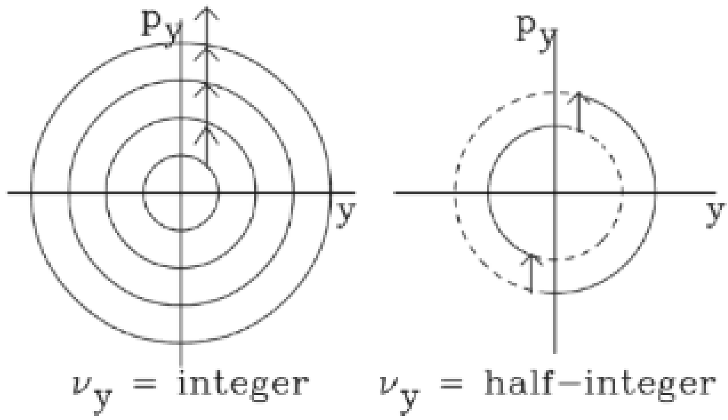

그림 8.1: For dipole errors, only integer tunes induce instability.

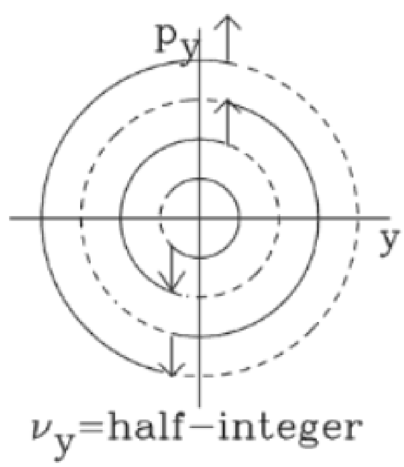

그림 8.2: For quadrupole errors, both half-integer and integer tunes induce instability.

- A. Wolski 책의 그림: 우리 교과서의 그림에서는 distorted closed orbit 이 미분값이 연속인 듯 그려진 것 같은데, 실제로는 연속이기는하나 미분값은 연속이지 않다.

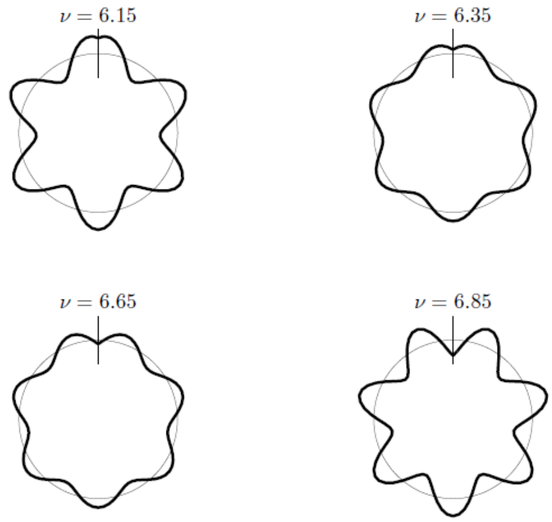

그림 8.3: Distortion of the closed orbit in a storage ring with different tune values. In each case, the reference trajectory is represented by a circle, with a straight line cutting the circumference at the location of a dipole 'kick'. The size of the dipole kick is the same for all cases. The closed orbit is represented by the distance of the heavy line from the reference trajectory. The amplitude of the closed orbit is largest for tune values close to an integer, and smallest for tune values close to a half integer.