# 제 7 장

# Action-angle variables for betatron oscillations

- In Chap. 3, we learned that choosing the action-angle canonical variables in 1D Hamiltonian system dramatically simplifies the dynamics: the action remains constant and the angle increases linearly with time.

- With minor modifications, the same transformation can be applied to the betatron oscillations in an accelerator.

- This yields an invariant of the motion and is also a useful starting point for analyzing more complicated dynamics.

## 제 1 절　Action-Angle Variables

- The general solution of the equations of motion:

$$x(s) = A\sqrt{\beta(s)}\cos\psi(s)\,, \tag{7.1}$$

where initial phase $\phi$ (or $\psi_0$) in now included into $\psi$ for simplicity. Note that the phase $\psi(s)$ is not necessarily a linear function of $s$.

- The canonical momentum:

$$\begin{aligned} P_x(s) = x'(s) &= \frac{A}{\sqrt{\beta}}\cos\psi(s)\left(\frac{\beta'}{2} - \tan\psi(s)\right) \\ &= \frac{x}{\beta}\left(\frac{\beta'}{2} - \tan\psi(s)\right)\,. \end{aligned} \tag{7.2}$$

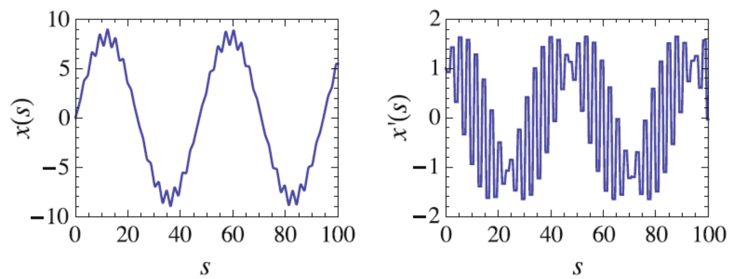


**Fig. 7.1** Plots of $x$ and $x'$ versus $s$ for a particular solution to Eq. (6.15) with initial conditions $x(0) = 0$ and $x'(0) = 1$

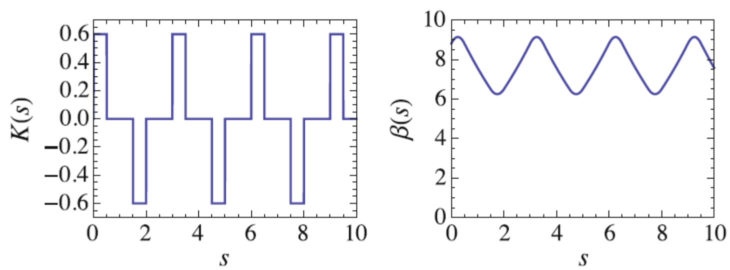


**Fig. 7.2** Function $K(s)$ (left panel) and the corresponding $\beta$ function (right panel)

그림 7.1: I think, in principle, $x'$ should be much smaller than 1.

- The same approach as in the case of the linear harmonic oscillator: assuming $A = A(J)$ and replacing $\psi$ by $\phi$.

$$x(s) = A(J)\sqrt{\beta(s)}\cos\phi\,, \tag{7.3}$$

$$P_x(s) = -\frac{x}{\beta(s)}\left(\alpha + \tan\phi\right)\,, \tag{7.4}$$

where we have introduced the following notation (so called *alpha-function*)

$$\alpha(s) = -\frac{1}{2}\beta'(s)\,. \tag{7.5}$$

- The generating function of the first kind to find the action (i.e., new canonical momentum)

$$P_x = \frac{\partial F_1}{\partial x}$$

$$F_1(x,\phi,s) = \int P_x dx = -\frac{x^2}{2\beta}\left(\alpha + \tan\phi\right)\,. \tag{7.6}$$

Therefore, we find the action ($d(\tan\phi)/d\phi = \sec^2\phi$)

$$J = -\frac{\partial F_1}{\partial\phi} = \frac{x^2}{2\beta}(1+\tan^2\phi)\,, \tag{7.7}$$

and from Eq. (7.4)[i.e., using $P_x = -(x/\beta)(\alpha + \tan\phi)$, we express $\phi$ in terms of $x$ and $P_x$]

$$\tan\phi = -\frac{\beta P_x}{x} - \alpha\,, \tag{7.8}$$

we obtain $J$ in terms of $x$ and $P_x$:

$$J = \frac{1}{2\beta}\left[x^2 + (\beta P_x + \alpha x)^2\right]\,. \tag{7.9}$$

Equations (7.8) and (7.9) give us the transformation $(x, P_x) \to (\phi, J)$.

- The inverse transformation $(\phi, J) \to (x, P_x)$:

$$J = \frac{x^2}{2\beta}(1 + \tan^2\phi) = \frac{x^2}{2\beta}\left(\frac{1}{\cos^2\phi}\right) \tag{7.10}$$

$$x = \sqrt{2\beta J}\cos\phi\,. \tag{7.11}$$

즉, 이식은 $A(J) = \sqrt{2J}$ 로 해석될 수 있다. 마찬가지로 $P_x$를 $J$ 와 $\phi$로 표현할 수 있다.

$$P_x = -\sqrt{\frac{2J}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\,. \tag{7.12}$$

- The new Hamiltonian:

$$\begin{aligned}\hat{\mathcal{H}} &= \mathcal{H} + \frac{\partial F_1}{\partial s} \\ &= \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2 + \frac{x^2}{4}\frac{\beta''\beta - \beta'^2}{\beta^2} + \frac{x^2\beta'}{2\beta^2}\tan\phi\,.\end{aligned} \tag{7.13}$$

After eliminating $\beta''$ using the envelope equation for $\beta$, and replacing $\tan\phi$ with $-\beta P_x/x - \alpha$,

$$\begin{aligned}\hat{\mathcal{H}} &= \frac{1}{2}P_x^2 + \frac{1}{2\beta^2}x^2 + \frac{\alpha^2}{2\beta^2}x^2 + \frac{\alpha}{\beta}P_x x \\ &= \frac{J}{\beta}\,.\end{aligned} \tag{7.14}$$

- Equation of motion for $J$:

$$J' = -\frac{\partial\hat{\mathcal{H}}}{\partial\phi} = 0\,, \tag{7.15}$$

which means that $J$ is an integral of motion. The quantity $2J$ is called the Courant-Snyder invariant.

- Equation of motion for $\phi$:

$$\phi' = \frac{\partial\hat{\mathcal{H}}}{\partial J} = \frac{1}{\beta(s)}\,. \tag{7.16}$$

Comparing with $\psi' = 1/\beta$, we see that the new coordinates $\phi$ is actually equal to the old betatron phase, $\phi = \psi + \text{const.}$

## 제 2 절 Eliminating Phase Oscillations

The phase coordinate $\phi$ monotonically grows with $s$, but with a rate of change that oscillates around some average value due to the oscillations of the beta function. We can do one more canonical transformation to straighten out these oscillations: $(\phi, J) \to (\phi_1, J_1)$

$$F_2(\phi, J_1, s) = J_1\left(\frac{2\pi\nu s}{C} - \int_0^s \frac{ds'}{\beta(s')}\right) + \phi J_1\,, \tag{7.17}$$

where $C$ is the circumference of the accelerator and $\nu$ is the tune.

- The new angle

$$\phi_1 = \frac{\partial F_2}{\partial J_1} = \phi + \frac{2\pi\nu s}{C} - \int_0^s \frac{ds'}{\beta(s')} = \phi + \frac{2\pi\nu s}{C} - \psi(s)\,, \tag{7.18}$$

- The action is unchanged,

$$J = \frac{\partial F_2}{\partial \phi} = J_1\,. \tag{7.19}$$

- The new Hamiltonian (actually, $J/\beta$ term in the old Hamiltonian is cancelled)

$$\hat{\mathcal{H}}_1 = \hat{\mathcal{H}} + \frac{\partial F_2}{\partial s} = \frac{2\pi\nu}{C}J_1\,. \tag{7.20}$$

- The equation of motion for $\phi_1$

$$\phi_1' = \frac{\partial \hat{\mathcal{H}}_1}{\partial J_1} = \frac{2\pi\nu}{C}\,, \tag{7.21}$$

  which means that $\phi$ is a linear function of $s$ with the slope given by $2\pi\nu/C$. Indeed, we got rid of the oscillations exhibited by the phase $\phi$ and obtained a new phase $\phi_1$ that follows a straight line in $s$. The tune is identical in both sets of coordinates.

## 제 3 절 Phase Space Motion at a Given Location

1. As a particle travels in a circular accelerator, every revolution period it arrives at the same longitudinal position $s$.

2. Let us consider the phase plane $(x, P_x)$ at this location $s$, and plot the particle coordinates every time it passes through $s$.

3. Because there is an integral of motion $J$, all these points are located on the curve $J = \text{const}$.

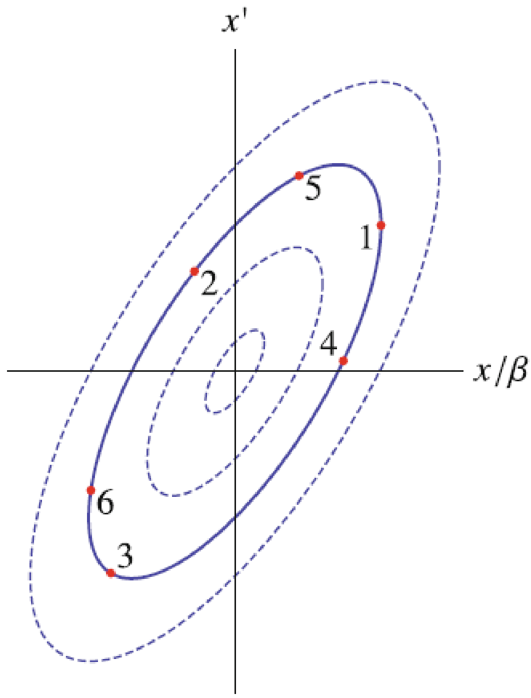


그림 7.2: The phase space ellipse (solid curve) and a particle's positions at consecutive turns. Dashed lines show ellipses for particles with smaller and larger values of action $J$. The vertical axis is marked by $x'$ which is equal to $P_x$. For convenience $x$ is normalized by the beta function at this location, $\beta(s)$.

4. Since
$$J = \frac{1}{2\beta}\left[x^2 + (\beta P_x + \alpha x)^2\right]$$
it follows that this curve is an ellipse whose size and orientation depend on the values of $J, \beta$, and $\alpha$.

5. Particles with different values of $J$ have geometrically similar ellipses enclosed inside each other.

6. If $\alpha = 0$, the ellipse turns into a circle. In this case, the trajectory is very simple: on each revolution the representative point rotates by the betatron phase advance in the ring $\Delta\psi = 2\pi\nu$ *in the clockwise direction.*

7. A set of ellipses at another location in the ring will have a different shape which is defined by the local values of $\beta$ and $\alpha$.

8. When one travels along the circumference of the accelerator, one sees a continuous transformation of these sets with the coordinate $s$.

9. For a collection of particles in a bunch, this effect includes changes not only in the size of the beam (e.g., $\langle x^2 \rangle$) but also in statistical correlations between $x$ and $P_x$ (e.g., $\langle x P_x \rangle$).