제 6 장

Equations of motion in accelerators

- A typical accelerator uses a sequence of various types of magnets separated by sections of free space (so-called *drifts*).
- To specify the Hamiltonian in Chap. 5, we need to know the vector potential A_s for these magnets.
- In this chapter, we assume that the field profiles are uniform over their length.
- Often in analysis and simulations, one has to take into account that at the end points of the magnets different field geometries appear, called *fringe fields*. The impact of these fields are usually treated as highly localized corrections which are calculated separately from the bulk of the magnet, and involve higher order terms.
- When fringe fields are weak they can be treated as field errors, which are covered in Chap. 8.

제 1 절 Vector potential for different types of magnets

In this section, we will list several magnet types and write down approximate expressions for $A_s(x, y)$. We are only interested in fields near the reference orbit, $|x|, |y| \ll |\rho|$, so we can neglect higher powers of the ratios x/ρ and y/ρ.

- *Dipole* magnets are used to bend the orbit and, in circular accelerators, to eventually make it close on itself.

$$\boldsymbol{B} = \hat{\boldsymbol{y}} B(s) \,. \tag{6.1}$$

Here, we assume that the field is directed along y and, in the lowest approximation, neglecting its variation in the transverse plane (that is, neglecting its dependence on x and y). The function $B(s)$ characterizes the longitudinal variation of the field, and vanishes outside of the magnets, and within the dipole the field can be represented by

$$A_s = -B(s)x\left(1 - \frac{x}{2\rho}\right) . \tag{6.2}$$

Using $\boldsymbol{B} = \nabla \times \boldsymbol{A}$,

$$\begin{aligned} B_y &= -\frac{1}{1+x/\rho}\frac{\partial A_s(1+x/\rho)}{\partial x} \\ &\approx B(s)\left(1-\frac{x}{\rho}\right)\frac{\partial}{\partial x}\left[x\left(1-\frac{x}{2\rho}\right)\left(1+\frac{x}{\rho}\right)\right] \\ &\approx B(s) + O\left(\frac{x^2}{\rho^2}\right) . \end{aligned} \tag{6.3}$$

또는, 좀더 정확하게 하려면 (Wolski 교과서 처럼),

$$A_s = -B(s)x\left(1 - \frac{x}{2\rho(1+x/\rho)}\right) . \tag{6.4}$$

$$\begin{aligned} B_y &= -\frac{1}{1+x/\rho}\frac{\partial A_s(1+x/\rho)}{\partial x} \\ &= B(s)\frac{1}{(1+x/\rho)}\frac{\partial}{\partial x}\left[x\left(1+\frac{x}{\rho}-\frac{x}{2\rho}\right)\right] \\ &= B(s)\frac{1}{(1+x/\rho)}\frac{\partial}{\partial x}\left[x\left(1+\frac{x}{2\rho}\right)\right] \\ &= B(s)\frac{1}{(1+x/\rho)}\times\left(1+\frac{x}{\rho}\right) \\ &= B(s) . \end{aligned} \tag{6.5}$$

- *Quadrupole* magnet is used to focus off-orbit particles so that they remain close to the reference orbit.

$$\boldsymbol{B} = G(s)(\hat{\boldsymbol{y}}x + \hat{\boldsymbol{x}}y) , \tag{6.6}$$

where the function $G(s)$ (often called field gradient) again isolates the longitudinal variation of the field. Note that the field on the axis is zero, which means that the reference orbit is a straight line ($\rho \to \infty$). The corresponding vector potential is

$$A_s = \frac{1}{2}G(s)\left(y^2 - x^2\right) . \tag{6.7}$$

- A *skew quadrupole* is a normal quadrupole rotated by 45°:

$$\boldsymbol{B} = G_{\rm sq}(s)(-\hat{\boldsymbol{y}}y + \hat{\boldsymbol{x}}x) , \tag{6.8}$$

with

$$A_s = G_{\rm sq}(s)xy . \tag{6.9}$$

- *Sextupole* magnets are used to correct some properties of the transverse properties of the transverse oscillations of the beam particles around the reference orbit (e.g., chromatic aberration correction). This magnet has a nonlinear (quadratic) dependence of the magnetic field with the transverse coordinates:

$$\boldsymbol{B} = S(s)\left[\frac{1}{2}\hat{\boldsymbol{y}}(x^2 - y^2) + \hat{\boldsymbol{x}}xy\right]. \tag{6.10}$$

  with the corresponding vector potential

$$A_s = S(s)\left(\frac{1}{2}xy^2 - \frac{1}{6}x^3\right). \tag{6.11}$$

  There is also a skew version of the sextupole.

제 2 절 Taylor expansion of the Hamiltonian

For circular accelerator that has dipole and quadrupole magnets, replacing A_s with the sum of the vector potentials (6.2) and (6.7):

$$\begin{aligned}\mathcal{H} &= -(1+\eta)\left(1+\frac{x}{\rho}\right)\left(1-\frac{1}{2}P_x^2-\frac{1}{2}P_y^2\right)\\ &\quad -\frac{e}{p_0}\left[-B(s)x\left(1-\frac{x}{2\rho}\right)+\frac{1}{2}G(s)\left(y^2-x^2\right)\right]\left(1+\frac{x}{\rho}\right)\\ &\approx -1-\eta-\eta\frac{x}{\rho}+\frac{1}{2}P_x^2+\frac{1}{2}P_y^2+\frac{x^2}{2\rho^2}-\frac{e}{p_0}\frac{1}{2}G(s)\left(y^2-x^2\right), \end{aligned}\tag{6.12}$$

1. We have made use of $\rho = \frac{p_0}{eB(s)}$.
2. Assuming η, x, P_x, y, P_y are of the first order, we neglected terms of the third and higher orders.
3. We will drop the constant terms such as -1 in the Hamiltonian, which has no significance for the dynamics.
4. We will mostly treat the case of on-momentum particles, $\eta = 0$.

With these assumptions, the Hamiltonian becomes the sum of two terms corresponding to the horizontal (x) and vertical (y) degrees of freedom as:

$$\mathcal{H} = \mathcal{H}_x + \mathcal{H}_y, \tag{6.13}$$

with

$$\mathcal{H}_x = \frac{1}{2}P_x^2 + \frac{x^2}{2\rho^2} + \frac{1}{2}\frac{e}{p_0}G(s)x^2 \tag{6.14}$$

and

$$\mathcal{H}_y = \frac{1}{2}P_y^2 - \frac{1}{2}\frac{e}{p_0}G(s)y^2. \tag{6.15}$$

- The fact that the Hamiltonian is split into two pieces each of which involves only variables corresponding to one degree of freedom means that the horizontal and vertical motion are *decoupled*.

- The skew quadrupole and sextupole magnets which have been left out of this example can in practice be used to correct unintended coupling as needed.

- The quadrupole magnetic field acts in opposite ways in x and y: positive G means that the effective potential energy in $\mathcal{H}_x$ has a minimum on axis $x = 0$ and leads to stable oscillations around this equilibrium point. At the same time, the effective potential energy in $\mathcal{H}_y$ has a maximum at $y = 0$, which is unstable.

- Negative G changes the sign of the effect in x and y. A sequence of quadrupoles with alternating polarities can make the transverse motion stable in both directions and confine it near the reference orbit. As a result, a particle near the equilibrium orbit executes stable *betatron* oscillations.

- Even in the absence of quadrupoles, there is a focusing force in the horizontal (x) direction inside dipole magnets: $x^2/(2\rho)$. Being inversely proportional to ρ^2, this term is typically small and does not play a big role in the beam dynamics (it is referred to as the *weak* focusing effect).

- To study general properties of the motion in both transverse planes, in the next section we will use a generic Hamiltonian

$$\mathcal{H}_0(x, P_x, s) = \frac{1}{2}P_x^2 + \frac{1}{2}K(s)x^2\,, \tag{6.16}$$

where

$$K = \frac{1}{\rho^2} + \frac{eG}{p_0} \text{ for the horizontal plane} \tag{6.17}$$

$$K = -\frac{eG}{p_0} \text{ for the vertical plane} \tag{6.18}$$

제 3 절 Hill's equation, betatron function and betatron phase

From the Hamiltonian (6.16),

$$x''(s) + K(s)x(s) = 0\,, \tag{6.19}$$

where the prime denotes the derivative with respect to s.

- In a circular accelerator, $K(s)$ is a periodic function of s with a period that we denote by L (which may be equal to the ring circumference or a fraction of it).

- Equation (6.19) with a periodic K is called Hill's equation; it describes the so-called *betatron* oscillations around the reference orbit.

- Note that we have encountered the same equation in the discussion of the parametric resonance, with the only difference that we now have s as an independent variable instead of t. We know that this equation can have both stable and unstable solutions.

3.1 Floquet transforamtion

Some of the fundamental properties of Eq. (6.19) can be studied without specifying the function $K(s)$. Let us seek its solution in the following form:

$$x(s) = Aw(s)\cos[\psi(s) + \phi]\,, \tag{6.20}$$

- Here, A and ϕ are arbitrary constants determined by the initial conditions.

$$x(0) = x_0 = Aw(0)\cos[\phi] \tag{6.21}$$

$$x'(0) = x'_0 = Aw'(0)\cos[\phi] - Aw(0)\psi'(0)\sin[\phi] \tag{6.22}$$

  두 개의 초기조건이 있어야, 2차 미분방정식의 해를 얻을 수 있으므로, 두 개의 arbitrary constants가 필요한 게 맞다. Here, we set $\psi(0) = 0$ without loss of generality (이 부분은 Stupakov의 책과 다른 notation인데, 나는 Ron Davidson과 Martin Reizer의 방식을 따른 것임).

- The two functions $w(s)$ and $\psi(s)$ are determined by the requirement that Eq. (6.20) satisfies Eq. (6.19).

- The function $w(s)$ is not uniquely defined: we can always multiply it by an arbitrary factor w_0 (i.e., $w \to w_0 w$) and redefine the amplitude $A \to A/w_0$, so that $x(s)$ and $x'(s)$ are not changed. 이것은 당연한 것임. 왜냐하면, 입자의 운동은 초기 조건 x_0와 x'_0에 의해 결정되기때문에, 어떤식으로 parametrization을 하느냐에 무관하다.

- If the particle motion is stable, we *can require* that $w(s)$ and $d\psi/ds$ be periodic functions of s with the period L. 이렇게 주기함수로 잡을 수 있다는 이야기이지 반드시 주기함수이여야 할 필요는 없다. 왜냐하면, 입자의 운동은 parametrization 방법에 무관하기 때문이다. 물론, 나중에 배우겠지만, $w(s)$와 $d\psi/ds$을 주기함수로 만드는 것이 matched beam을 만들어 emittance 증가가 최소화 된다.

- Periodicity in $d\psi/ds$ can equivalently formulated as

$$\psi(s+L) - \psi(s) = \int_s^{s+L} \frac{d\psi}{ds}(s')ds' \equiv \sigma = \text{const. independent of } s \tag{6.23}$$

  Here, σ is called *phase advance* per period.

- Introducing the two unknown functions $w(s)$ and $\psi(s)$ instead of one $x(s)$ gives us the freedom to impose a constraint of our choice on the functions w and ψ to obtain an optimal parametrization of the solution.

- Substituting Eq. (6.20) into (6.19) we obtain

$$\big[w'' - w\psi'^2 + K(s)w\big]\cos[\psi(s)+\phi] - (2w'\psi' + w\psi'')\sin[\psi(s)+\phi] = 0\,. \quad (6.24)$$

위의 식이 초기 phase ϕ 에 무관하게 항상 성립하려면,

$$\begin{aligned} w'' - w\psi'^2 + K(s)w &= 0\,, \\ -2w'\psi' - w\psi'' &= 0\,. \end{aligned} \quad (6.25)$$

3.2 Phase advance rate

- 위의 마지막 식에서부터

$$\frac{1}{w}(\psi' w^2)' = 0\,. \quad (6.26)$$

We integrate it and introduce the *beta function*, $\beta(s) = w^2(s)$

$$\psi' = \frac{a}{\beta(s)}\,, \quad (6.27)$$

where a is an arbitrary constant of integration.

- Without loss of generality, we can assume that $a > 0$; if this is not the case, we can always change its sign by redefining the angle $\psi \to -\psi$, which does not change $x(s)$. As was pointed out above, the function w can be multiplied by an arbitrary constant factor. Choosing this factor equal to $\sqrt{a}$ and replacing $\beta \to a\beta$ eliminates a.

$$\psi' = \frac{1}{\beta(s)}\,. \quad (6.28)$$

3.3 Envelope equation

- 또 다른 방정식으로부터

$$w'' - \frac{1}{w^3} + K(s)w = 0\,. \quad (6.29)$$

This equation is called *betatron envelope equation*. By substituting $w(s) = \sqrt{\beta} > 0$ (초기에 w 가 양수이면, 0으로 줄어들다가도 w^{-3} 항 때문에 다시 커진다. 따라서, 한번 양수이면 계속 양수이고, 마찬가지로 한번 음수이면 계속 음수이다. Since the sign of the betatron function is not determined and does not change, it has become customary to use only the positive solution), we obtain

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2 = 1\,. \quad (6.30)$$

This is a nonlinear differential equation of the second order for $\beta(s)$.

- For a given periodic function $K(s)$, we can solve the envelope equation for given initial values of β and β'. We can impose the same periodicity condition on the beta function.

$$\beta(s+L) = \beta(s), \quad \beta'(s+L) = \beta'(s) \tag{6.31}$$

After $\beta(s)$ is found, the betatron phase ψ is obtained by a straightforward integration. For a periodic $\beta(s)$ the derivative ψ' is also periodic with the same period L.

3.4 Tune in a ring

An important characteristic of the magnetic lattice of ring is the betatron phase advance over its circumstance C:

$$\Delta\psi = \int_0^C \frac{ds}{\beta(s)} \tag{6.32}$$

The quantity $\Delta\psi/(2\pi)$ is called the *tune* ν (also denoted by Q in the European literature),

$$\nu = \frac{1}{2\pi} \int_0^C \frac{ds}{\beta(s)} \,. \tag{6.33}$$

It is the number of transverse oscillations that a particle makes as it circulates once around the ring. As we will see in the following chapters, the tune plays an important role in beam dynamics.

Notes

- The main advantage of introducing the two functions $\beta(s)$ and $\psi(s)$ is that, for a given magnetic lattice, they need to be calculated only once. Having found them, the general solution to the equation of motion can be written as

$$x(s) = A\sqrt{\beta(s)}\cos[\psi(s) + \phi] \,, \tag{6.34}$$

where A and ϕ are two arbitrary constants that depend on the initial conditions.

- Note that the phase term only needs to be adjusted by a constant offset for different initial conditions.

- Even without detailed knowledge of initial conditions (i.e., A's and ϕ's), this equation gives important information about the structure of particle trajectories in the ring.

- We mention that although our analysis was motivated by circular accelerators, the same representation (6.34) of particle orbits is often used in linear accelerators.

- In the absence of the periodicity condition in such machines, to solve envelope equation in β one needs either to specify the initial β and its derivative β' at the entrance to the system, or to impose equivalent boundary conditions.