제 5 장

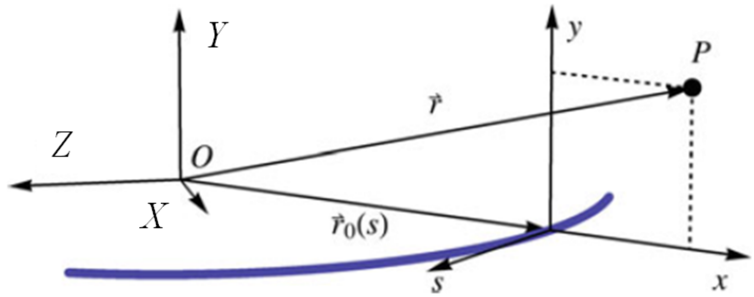
Coordinate system and Hamiltonian for a circular accelerator

Assumptions:

- First, we assume that there is no electrostatic field, $\phi = 0$, and the magnetic field does not vary with time.
- Second, the magnetic field is arranged in such a way that there is a closed reference (or nominal) orbit for a particle with a nominal momentum p_0—this is achieved by a proper design of the magnetic lattice of the ring.
- We will also assume that this reference orbit is a plane curve lying in the horizontal plane.
- Our goal is to describe the motion in the vicinity of this reference orbit of particles having energies (or, equivalently, momenta) that can slightly deviate from the nominal one.

제 1 절 Coordinate System

- A segment of the reference orbit is specified by the vector function $\boldsymbol{r}_0(s)$.
- s is the arclength measured along the reference orbit in the direction of motion.
- Three unit vectors for the local coordinate system:
  - Tangential to the orbit: $\hat{\boldsymbol{s}} = d\boldsymbol{r}_0/ds$
  - Perpendicular to $\hat{\boldsymbol{s}}$ and in the plane of the orbit: $\hat{\boldsymbol{x}}$
  - Perpendicular to the plane of the orbit: $\hat{\boldsymbol{y}} = \hat{\boldsymbol{s}} \times \hat{\boldsymbol{x}}$



- The three unit vectors, being defined locally, vary with s:

$$\frac{d\hat{\boldsymbol{s}}}{ds} = -\frac{\hat{\boldsymbol{x}}}{\rho(s)}\,, \tag{5.1}$$

$$\frac{d\hat{\boldsymbol{x}}}{ds} = \frac{\hat{\boldsymbol{s}}}{\rho(s)}\,, \tag{5.2}$$

$$\frac{d\hat{\boldsymbol{y}}}{ds} = 0\,, \tag{5.3}$$

  where $\rho(s)$ is the radius of curvature of the reference orbit.

  - Since we have made the simplifying assumption that there is *no vertical bending*, $\frac{d\hat{\boldsymbol{x}}}{ds}$ is always parallel to the longitudinal coordinate $\hat{\boldsymbol{s}}$ and the reference orbit lies in a plane.
  - In the more general case of orbits that move out of a single plane, the expressions for the derivatives above would have additional terms related to *torsion*, and the dipole magnetic field could also have an x component, but those will not be considered here. Under the above constraints, the torsion is always zero.
  - The absolute value of the radius of curvature ρ is given by the equation

$$|\rho(s)| = \frac{p_0}{|eB_y(s)|}\,, \tag{5.4}$$

    where p_0 is the kinetic momentum of the reference particle and B_y is the vertical component of the dipole magnetic field. The $\rho(s)$ is generally different from $|\boldsymbol{r}_0(s)|$ (for example, consider a straight section in the ring where $\rho \to \infty$).

- Beam particle deviate from the reference orbit, but move close to it. A particle position is represented by

$$\boldsymbol{r} = \boldsymbol{r}_0(s) + x\hat{\boldsymbol{x}}(s) + y\hat{\boldsymbol{y}}\,. \tag{5.5}$$

$$\begin{aligned}
\frac{d\boldsymbol{r}}{ds} &= \frac{d\boldsymbol{r}_0}{ds} + \frac{dx}{ds}\hat{\boldsymbol{x}} + x\frac{d\hat{\boldsymbol{x}}}{ds} + \frac{dy}{ds}\hat{\boldsymbol{y}} + \cancel{y\frac{d\hat{\boldsymbol{y}}}{ds}} \\
&= \hat{\boldsymbol{s}} + \frac{dx}{ds}\hat{\boldsymbol{x}} + x\frac{\hat{\boldsymbol{s}}}{\rho(s)} + \frac{dy}{ds}\hat{\boldsymbol{y}}
\end{aligned}$$

Therefore,

$$d\boldsymbol{r} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + [1 + x/\rho(s)]ds\hat{\boldsymbol{s}} = h_1\boldsymbol{u}_1 du_1 + h_2\boldsymbol{u}_2 du_2 + h_3\boldsymbol{u}_3 du_3 \tag{5.6}$$

The scale factors in the orthogonal curvilinear coordinates are

$$h_1 = 1, \quad h_2 = 1, \quad h_3 = 1 + x/\rho(s) \tag{5.7}$$

- From the formulae for differential operators in orthogonal curvilinear coordinates:

$$\nabla\phi = \hat{\boldsymbol{x}}\frac{\partial\phi}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial\phi}{\partial y} + \hat{\boldsymbol{s}}\frac{1}{1+x/\rho}\frac{\partial\phi}{\partial s}\,, \tag{5.8}$$

$$(\nabla\times\boldsymbol{A})_x = -\frac{1}{1+x/\rho}\frac{\partial A_y}{\partial s} + \frac{\partial A_s}{\partial y}\,, \tag{5.9}$$

$$(\nabla\times\boldsymbol{A})_y = -\frac{1}{1+x/\rho}\frac{\partial A_s(1+x/\rho)}{\partial x} + \frac{1}{1+x/\rho}\frac{\partial A_x}{\partial s}\,, \tag{5.10}$$

$$(\nabla\times\boldsymbol{A})_s = -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\,, \tag{5.11}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{1+x/\rho}\frac{\partial A_x(1+x/\rho)}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{1}{1+x/\rho}\frac{\partial A_s}{\partial s}\,. \tag{5.12}$$

where

$$\boldsymbol{A} = A_x(x,y,s)\hat{\boldsymbol{x}} + A_y(x,y,s)\hat{\boldsymbol{y}} + A_s(x,y,s)\hat{\boldsymbol{s}} \tag{5.13}$$

제 2 절 Hamiltonian in Curvilinear Coordinate System

- The general Hamiltonian for a charged particle with $\phi = 0$:

$$H = \sqrt{(mc^2)^2 + c^2(\boldsymbol{\pi} - e\boldsymbol{A})^2}\,. \tag{5.14}$$

This was derived for a Cartesian coordinate system (X, Y, Z), which are the old (original) coordinates.

- To transform from old (original) coordinates to new local coordinates (x, y, s), we use the generating function of the third type:

$$F_3(\boldsymbol{\pi}, x, y, s) = -\boldsymbol{\pi}\cdot(\boldsymbol{r}_0(s) + x\hat{\boldsymbol{x}}(s) + y\hat{\boldsymbol{y}})\,. \tag{5.15}$$

Here, $\boldsymbol{\pi}$ is the old momentum.

- The new canonical momentum $\boldsymbol{\Pi}$ is

$$\Pi_x = -\frac{\partial F_3}{\partial x} = \boldsymbol{\pi}\cdot\hat{\boldsymbol{x}} = \pi_x\,,$$

$$\Pi_y = -\frac{\partial F_3}{\partial y} = \boldsymbol{\pi}\cdot\hat{\boldsymbol{y}} = \pi_y\,,$$

$$\Pi_s = -\frac{\partial F_3}{\partial s} = \boldsymbol{\pi}\cdot\left(\frac{d\boldsymbol{r_0}}{ds} + x\frac{d\hat{\boldsymbol{x}}}{ds}\right) = \boldsymbol{\pi}\cdot\left(\hat{\boldsymbol{s}} + \frac{x}{\rho}\hat{\boldsymbol{s}}\right) = \pi_s\left(1 + \frac{x}{\rho}\right)\,, \tag{5.16}$$

- Since

$$(\boldsymbol{\pi} - e\boldsymbol{A})^2 = (\pi_x - eA_x)^2 + (\pi_y - eA_y)^2 + (\pi_s - eA_s)^2$$
$$= (\Pi_x - eA_x)^2 + (\Pi_y - eA_y)^2 + \left(\frac{\Pi_s}{1+x/\rho} - eA_s\right)^2 , \tag{5.17}$$

our new Hamiltonian can be written as

$$H = c\left[m^2c^2 + (\Pi_x - eA_x)^2 + (\Pi_y - eA_y)^2 + \left(\frac{\Pi_s}{1+x/\rho} - eA_s\right)^2\right]^{1/2} . \tag{5.18}$$

Here, $A_x = \boldsymbol{A}\cdot\hat{\boldsymbol{x}}$, $A_y = \boldsymbol{A}\cdot\hat{\boldsymbol{y}}$, $A_s = \boldsymbol{A}\cdot\hat{\boldsymbol{s}}$.

제 3 절 Using s as Time Variable

For a time-independent Hamiltonian, Hamiltonian is a constant of motion of value h:

$$h = H(x, \Pi_x, y, \Pi_y, s, \Pi_s)\,, \tag{5.19}$$

Solving it, we find Π_s:

$$\Pi_s = \Pi_s(x, \Pi_x, y, \Pi_y, h, s)\,, \tag{5.20}$$

We introduce a new Hamiltonian K as

$$K(x, \Pi_x, y, \Pi_y, h, s) = -\Pi_s(x, \Pi_x, y, \Pi_y, h, s)\,. \tag{5.21}$$

Here, x, Π_x, y, Π_y are considered as canonical conjugate variables, s is an independent *time-like* variable, and h is a constant parameter.

3.1 Proof

- Using the old Hamiltonian H, we have

$$\frac{dx}{ds} = \frac{dx/dt}{ds/dt} = \frac{\partial H/\partial\Pi_x}{\partial H/\partial\Pi_s}\,, \tag{5.22}$$

The derivative $\partial K/\partial\Pi_x$ can be calculated as a derivative of an implicit function by differentiating h with respect to Π_x,

$$\frac{\partial h}{\partial\Pi_x} = 0 = \frac{\partial H}{\partial\Pi_x} + \frac{\partial H}{\partial\Pi_s}\left(\frac{\partial\Pi_s}{\partial\Pi_x}\right)_h \tag{5.23}$$

$$\frac{\partial K}{\partial\Pi_x} = -\left(\frac{\partial\Pi_s}{\partial\Pi_x}\right)_h = \frac{\partial H/\partial\Pi_x}{\partial H/\partial\Pi_s}\,. \tag{5.24}$$

Therefore,

$$\frac{dx}{ds} = \frac{\partial K}{\partial\Pi_x}\,, \tag{5.25}$$

which is a Hamiltonian equation for dx/ds.

- Similarly,

$$\frac{d\Pi_x}{ds} = \frac{d\Pi_x/dt}{ds/dt} = \frac{-\partial H/\partial x}{\partial H/\partial \Pi_s}\,. \tag{5.26}$$

$$\frac{\partial h}{\partial x} = 0 = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial \Pi_s}\left(\frac{\partial \Pi_s}{\partial x}\right)_h \tag{5.27}$$

$$\frac{\partial K}{\partial x} = -\left(\frac{\partial \Pi_s}{\partial x}\right)_h = \frac{\partial H/\partial x}{\partial H/\partial \Pi_s}\,. \tag{5.28}$$

Therefore,

$$\frac{d\Pi_x}{ds} = -\frac{\partial K}{\partial x} \tag{5.29}$$

which is a Hamiltonian equation for $d\Pi_x/ds$.

- Although time is now eliminated from the equations, the time dependence of s can be easily recovered. In the original Hamiltonian equations,

$$\frac{ds}{dt} = \frac{\partial H}{\partial \Pi_s} \tag{5.30}$$

$$\frac{dh}{dh} = 1 = \frac{\partial H}{\partial \Pi_s}\frac{\partial \Pi_s}{\partial h} \tag{5.31}$$

Therefore,

$$\frac{dt}{ds} = \frac{1}{\partial H/\partial \Pi_s} = \frac{\partial \Pi_s}{\partial h} = \frac{\partial K}{\partial(-h)}\,. \tag{5.32}$$

which is a Hamiltonian equation for $(t, -h)$ conjugate pair. Integrating this equation over s we find $t = t(s)$ with the inverse function defining $s(t)$.

제 4 절 Small Amplitude Approximation

- Solving Eq. (5.18) for Π_s,

$$\begin{aligned} K = -\Pi_s = &-\left(1+\frac{x}{\rho}\right)\left[\frac{1}{c^2}h^2 - (\Pi_x - eA_x)^2 - (\Pi_y - eA_y)^2 - m^2c^2\right]^{1/2} \\ &- eA_s\left(1+\frac{x}{\rho}\right)\,. \end{aligned} \tag{5.33}$$

- In many cases (except for solenoidal magnets), $A_x = A_y = 0$:

$$\Pi_x = p_x = \gamma m v_x, \quad \Pi_y = p_y = \gamma m v_y \tag{5.34}$$

$$K = -\left(1+\frac{x}{\rho}\right)\left(\frac{1}{c^2}h^2 - p_x^2 - p_y^2 - m^2c^2\right)^{1/2} - eA_s\left(1+\frac{x}{\rho}\right)\,. \tag{5.35}$$

- Particles in an accelerator beam typically move at small angles relatives to the reference orbit. This means p_x and p_y are small in comparison with the total momentum p, and the square root in Eq. (5.35) can be expanded as a Taylor series.

$$K \approx -p\left(1+\frac{x}{\rho}\right)\left(1-\frac{p_x^2}{2p^2}-\frac{p_y^2}{2p^2}\right)-eA_s\left(1+\frac{x}{\rho}\right), \tag{5.36}$$

where

$$E^2 = p^2c^2 + (mc^2)^2 = h^2 \to p(h) = \sqrt{h^2/c^2 - m^2c^2} \tag{5.37}$$

- It is convenient to introduce dimensionless quantities $P_x = p_x/p_0 \ll 1$ and $P_y = p_y/p_0 \ll 1$, where p_0 is the nominal momentum in the ring. The transformation from x, p_x, y, p_y to x, P_x, y, P_y is not canonical (not phase-space preserving), but it does not change the Hamiltonian structure.

$$\begin{aligned}\mathcal{H}(x,P_x,y,P_y) &= \frac{K}{p_0} \qquad (5.38)\\ &= -\frac{p}{p_0}\left(1+\frac{x}{\rho}\right)\left[1-\frac{1}{2}P_x^2\left(\frac{p_0}{p}\right)^2-\frac{1}{2}P_y^2\left(\frac{p_0}{p}\right)^2\right]-\frac{e}{p_0}A_s\left(1+\frac{x}{\rho}\right).\end{aligned}$$

- 특정 입자의 모멘텀 (또는 에너지)가 기준 값에서 살짝 벗어나 있는 경우:

$$\frac{p}{p_0} = 1+\eta, \tag{5.39}$$

where $\eta \ll 1$. 여기서는 η 를 일단 일정하다고 가정하지만, time-dependent 의 경우 η 도 변수가 된다.

$$\begin{aligned}&\mathcal{H}(x,P_x,y,P_y) \qquad (5.40)\\ &\approx -(1+\eta)\left(1+\frac{x}{\rho}\right)\left(1-\frac{1}{2}P_x^2-\frac{1}{2}P_y^2\right)-\frac{e}{p_0}A_s\left(1+\frac{x}{\rho}\right),\end{aligned}$$

- 마지막으로 $(P_x, P_y) \approx (x', y')$ 적용.

$$x' \equiv \frac{dx}{ds} = \frac{v_x}{v_s} = \frac{p_x}{p_s} \approx P_x, \tag{5.41}$$

제 5 절 Time-Dependent Hamiltonian

- 3절에서 time-independent Hamiltonian 을 가정하였는데, 실제로는 그렇게 하지 않아도 됨. 물론 이 경우, 에너지(h)가 일정하지는 않으므로, 원래 3개의 자유도가 그대로 유지됨.

$$h = K(x, \Pi_x, y, \Pi_y, s, \Pi_s, t) \neq \text{const.} \tag{5.42}$$

The new Hamiltonian becomes a function of time:

$$K(x,\Pi_x,y,\Pi_y,t,h,s) = -\Pi_s(x,\Pi_x,y,\Pi_y,t,h,s). \tag{5.43}$$

From Eq. (5.44), we have a Hamiltonian equation for $t(s)$, if $-h$ is associated with the momentum conjugate to t.

$$\frac{dt}{ds} = \frac{1}{\partial H/\partial \Pi_s} = \frac{\partial \Pi_s}{\partial h} = \frac{\partial K}{\partial(-h)}\,. \tag{5.44}$$

이렇게 $(t, -h)$가 conjugate pair가 되려면,

$$\frac{d(-h)}{ds} = -\frac{\partial K}{\partial t}. \tag{5.45}$$

도 만족을 해야함.

- 식 (5.45)의 증명:

$$\frac{dh}{ds} = \frac{dh/dt}{ds/dt} = \frac{\partial H/\partial t}{\dot{s}} \tag{5.46}$$

Here, we used the Poisson bracket relation

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \cancel{\{H, H\}} \tag{5.47}$$

Further, we have (이 식 자체가 어떻게 나왔는지 잘 증명이 안됨: 중간고사 숙제)

$$\frac{\partial H}{\partial t} = -\frac{\partial K/\partial t}{\partial K/\partial h} \tag{5.48}$$

위의 세식을 적당히 조합하면, 식 (5.45)이 나옴.

- Time-dependent Hamiltonian 이 필요한 예: Description of acceleration of charged particle by RF electromagnetic fields.
  - We assume the field is localized in a short RF cavity which is powered to voltage V.
  - An additional term needs to be added to the Hamiltonian K.

$$\frac{eV}{\omega_{\mathrm{RF}}}\delta(s - s_0)\sin(\omega_{\mathrm{RF}} t + \phi)\,, \tag{5.49}$$

  - Change in the kinetic energy after a passage through the point s_0 at time t:

$$\frac{dh}{ds} = \frac{\partial K}{\partial t} = eV\delta(s - s_0)\cos(\omega_{\mathrm{RF}} t + \phi) \tag{5.50}$$

$$\Delta h = eV\cos(\omega_{\mathrm{RF}} t + \phi) \tag{5.51}$$

$$(\Delta h)_{\max} = eV \tag{5.52}$$

제 6 절　보충: Wolski 책 내용

- 실제로 time-dependent 한 경우 한 가지 스텝이 더 필요함.
- "The next step is to define new longitudinal variables. The problem with the current variables $(t, -E)$ [*이 수업의 교과서 Stupakov 책의 notation으로 하면,* $(t, -h)$] is that the 'co-ordinate' (actually the time t) increases with distance down the beam line. In beam dynamics, we are often mostly concerned with the relative distance between two particles in the beam: this distance can be smaller than the length of the beam line by many orders of magnitude. Therefore, if we try to calculate the relative position of two particles after tracking them down a long beam line, we need to take the difference of two large numbers that are almost equal. In practice, it can be very difficult to maintain good accuracy in the calculations using this procedure. Instead, we shall define **a new longitudinal co-ordinate that has the physical significance of a distance relative to a nominal particle** that moves down the beam line with the reference momentum P_0 [*이 수업의 교과서 Stupakov 책의 notation으로 하면,* p_0]. If all particles in the beam have momentum close to P_0, then we should expect that the new longitudinal co-ordinate should remain conveniently small."

제 7 절　숙제: Orthogonal Curvilinear Coordinates

새로운 좌표계 (u_1, u_2, u_3)와 각각에 해당하는 scale factor가 (h_1, h_2, h_3)로 주어진 경우, 일반적으로 쓸 수 있는 식 조사할 것.

1. $\nabla\phi =$
2. $\nabla \times \boldsymbol{A} =$
3. $\nabla \cdot \boldsymbol{A} =$