제 4 장

Linear and Nonlinear Oscillators

The linear oscillator is a simple model that lies at the foundation of many physical phenomena and plays a crucial role in accelerator dynamics. Many systems can be viewed as an approximation to a set of independent linear oscillators.

제 1 절 Harmonic Oscillator Without and With Damping

- Ideal harmonic oscillator without damping:

$$x(t) = a\cos(\omega_0 t + \phi_0)\,, \tag{4.1}$$

- Damping due to a friction force that is proportional to the velocity $\dot{x}$:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0\,, \tag{4.2}$$

  where where γ is the damping constant and has the dimension of frequency. When the damping is not too strong, $\gamma < 2\omega_0$, (어떤 책에서는 2γ 로 쓰이니 주의할 것.)

$$x(t) = ae^{-\gamma t/2}\cos(\omega_1 t + \phi_0)\,, \tag{4.3}$$

  with

$$\omega_1 = \omega_0\sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}\,. \tag{4.4}$$

  If $\gamma \ll \omega_0$, the frequency ω_1 is close to ω_0, $\omega_1 \approx \omega_0$.

- The damping effect is often quantified by the so-called *quality factor* Q defined as

$$Q = 2\pi\frac{E}{|\Delta E|} = \frac{\omega_0}{\gamma}$$

where,

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2$$

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2\right] = -\gamma\dot{x}^2$$

$$\Delta E = -\int_{1\text{ period}} \frac{dE}{dt}dt \approx \pi\omega_0^2 a^2 \frac{\gamma}{\omega_0}$$

The regime of weak damping is characterized by $Q \gg 1$.

- If the oscillator is driven by an external force

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = f(t)\,, \tag{4.5}$$

where $f(t)$ is the force divided by the oscillator mass. The general solution for $\gamma = 0$:

$$x(t) = x_0 \cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t + \frac{1}{\omega_0}\int_0^t \sin[\omega_0(t-t')]f(t')dt'\,. \tag{4.6}$$

The general solution for $\gamma \neq 0$:

$$x(t) = e^{-\gamma t/2}\left[x_0\cos\omega_1 t + \left(\frac{x_0\gamma}{2\omega_1} + \frac{\dot{x}_0}{\omega_1}\right)\sin\omega_1 t\right] + \frac{1}{\omega_1}\int_0^t e^{-\gamma(t-t')/2}\sin\left[\omega_1(t-t')\right]f(t')dt'\,.$$

Here, $x_0 = x(0)$ and $\dot{x}_0 = \dot{x}(0)$. In both cases, Green functions (for $t \geq t'$) are used respectively:

$$G(t,t') = \frac{1}{\omega_0}\sin[\omega_0(t-t')],\quad G(t,t') = \frac{1}{\omega_1}e^{-\gamma(t-t')/2}\sin\left[\omega_1(t-t')\right]$$

보충: 3.5 Non-conservative Forces in Hamiltonian Dynamics

Damping and arbitrary externally-applied forces lead to equations of motion that do not quite match the framework of the Hamiltonian and Lagrangian formalisms. We can consider such terms to be corrections to the equations of motion.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_i\,, \qquad i = 1,\ldots,n\,, \tag{4.7}$$

where $F_i = F_i(q_k, \dot{q}_k)$ is a generalized force.

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + F_i\,,$$
$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\,.$$

In these equations F_i is now understood as a function of the Hamiltonian variables q_k and p_k which is obtained by expressing $\dot{q}_k$ in the arguments of F_i through these variables.

$$\frac{df}{dt}_{\text{conv}} = \frac{\partial f}{\partial t} + \{f, H\} + \sum_i F_i \frac{\partial f}{\partial p_i} \,.$$

Sometimes this force can be defined in terms of a potential-like term $R(q_k, \dot{q}_k)$, called the Rayleigh dissipation function, as

$$F_i = -\frac{\partial R}{\partial \dot{q}_i}$$

Although R does not represent a true potential or relate to any conserved quantity, it is convenient because, in contrast to F_i, it does not change under coordinate transformations (? 나중에 다시).

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i F_i \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial t} + \sum_i F_i \dot{q}_i = \frac{\partial H}{\partial t} - \sum_i \dot{q}_i \frac{\partial R}{\partial \dot{q}_i} \,,$$

where the last expression is for a frictional force corresponding to a potential R, but the derivatives of this potential and the $\dot{q}_i$ terms should be viewed as functions of the q_i and p_i coordinates.

- Harmonic oscillator with friction with $p = \dot{x}$:

$$F = -\gamma \dot{x} = -\gamma p \tag{4.8}$$

$$R = \gamma p^2/2 \tag{4.9}$$

$$\frac{dH}{dt} = -\gamma p^2 \tag{4.10}$$

For a quadratic potential, in the absence of damping, the average of H over one period is

$$\left\langle \frac{1}{2} p^2 \right\rangle + \left\langle \frac{1}{2} \omega_0^2 x^2 \right\rangle = \langle p^2 \rangle = \langle H \rangle = H \tag{4.11}$$

For weak damping, we find the approximate relation

$$\frac{d}{dt} \langle H \rangle \simeq -\gamma \langle H \rangle \,, \tag{4.12}$$

so the Hamiltonian decays as

$$\langle H \rangle \simeq H_0 e^{-\gamma t} \tag{4.13}$$

This is also consistent with the definition of quality factor

$$Q = \frac{2\pi}{T_0} \frac{E}{|\Delta E|/T_0} = \frac{\omega_0}{\gamma} \to \frac{|\Delta E|/T_0}{E} \sim \gamma = \frac{\omega_0}{Q}$$

제 2 절 Resonance in a Damped Oscillator

- An oscillator driven by a sinusoidal force with frequency ω: $f(t) = f_0 \cos \omega t$

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = f(t) = f_0 \cos \omega t$$

It is convenient to represent $x(t)$ as the real part of a complex function $\xi(t)$, $x(t) = \mathrm{Re}[\xi(t)]$.

$$\ddot{\xi} + \gamma \dot{\xi} + \omega_0^2 \xi = f_0 e^{-i\omega t}\,, \tag{4.14}$$

Taking real part gives the original equation.

- A particular solution in the form of $\xi(t) = \xi_0 e^{-i\omega t}$, where $\xi_0 = |\xi_0| e^{i\phi_0}$ is a complex number (a phasor):

$$x(t) = \mathrm{Re}[|\xi_0| e^{-i\omega t + i\phi_0}] = |\xi_0| \cos(\omega t - \phi_0) \tag{4.15}$$

$$\xi_0 = \frac{f_0}{\omega_0^2 - \omega^2 - i\omega\gamma}\,. \tag{4.16}$$

$$|\xi_0|^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\,. \tag{4.17}$$

When $\gamma \ll \omega_0$, $|\xi_0|^2$ exhibits *resonant* behavior: the amplitude of the oscillation increases when the driving frequency approaches the oscillation frequency ω_0. The *resonance width* $\Delta\omega_{\mathrm{res}}$ is defined as a characteristic width of the resonant curve. A crude estimate is

$$\Delta\omega_{\mathrm{res}} \sim \gamma \sim \frac{\omega_0}{Q} \tag{4.18}$$

제 3 절 Random Kicks

What happens if the external force is a sequence of random kicks?

- Let us assume that the external force is given by the following expression:

$$f(t) = \sum_i f_i \delta(t - t_i)\,, \tag{4.19}$$

where t_i are random moments of time, and the kick amplitudes f_i take random variable with zero average value, $\langle f_i \rangle = 0$.

- The forma solution with $\gamma = 0$ is

$$x(t) = \frac{1}{\omega_0} \int_0^t \sin[\omega_0(t - t')] f(t') dt' = \sum_i \frac{f_i}{\omega_0} \sin[\omega_0(t - t_i)]\,, \tag{4.20}$$

where we have assumed that at a time $t = 0$ the oscillator was at rest, $x_0 = \dot{x}_0 = 0$.

- It makes sense to consider the amplitude squared a^2 *averaged* over many random realization of the random force with the same statistical properties. Since

$$a^2 = x^2 + \omega_0^{-2}\dot{x}^2 \tag{4.21}$$

we consider the following quantity:

$$\begin{aligned}
&\langle x(t)^2 + \omega_0^{-2}\dot{x}^2(t)\rangle = \\
&= \omega_0^{-2}\sum_{i,j}\langle f_i f_j \left\{\sin[\omega_0(t-t_i)]\sin[\omega_0(t-t_j)] + \cos[\omega_0(t-t_i)]\cos[\omega_0(t-t_j)]\right\}\rangle \\
&= \omega_0^{-2}\sum_{i,j}\langle f_i f_j \cos[\omega_0(t_i - t_j)]\rangle\,,
\end{aligned} \tag{4.22}$$

Let us assume that t_i and t_j are statistically independent random numbers, and they are not correlated with the kick amplitudes f_i. Then the averaging of $f_i f_j$ can be split from the averaging of the cosine functions. Using $\langle f_i f_j\rangle = f_{\rm rms}^2\delta_{ij}$,

$$\langle x(t)^2 + \omega_0^{-2}\dot{x}^2(t)\rangle = \frac{f_{\rm rms}^2}{\omega_0^2}N(t)\,, \tag{4.23}$$

where $N(t)$ is the average number of kicks in the interval $[0, t]$. It can be estimated as $N(t) \approx t/\Delta t$, where δt is the averaged time between kicks.

- Notes:
  - The square of the oscillation amplitude grows linearly with time which is a characteristic feature of the diffusion process.
  - Hence the random uncorrelated kicks lead to diffusion-like behavior of the oscillation amplitude with time.
  - In the limit of $\omega_0 \to 0$, which corresponds to a free particle, we obtain

    $$\langle \dot{x}^2(t)\rangle = f_{\rm rms}^2 N(t)\,. \tag{4.24}$$

    This is a well known result for the velocity diffusion of a free particle caused by uncorrelated random kicks.

제 4 절 Parametric Resonance and Slow Variation of the Oscillator Parameters

What happens if $\omega_0(t)$ is a periodic function of time?

$$\ddot{x} + \omega_0^2(t)x = 0\,, \tag{4.25}$$

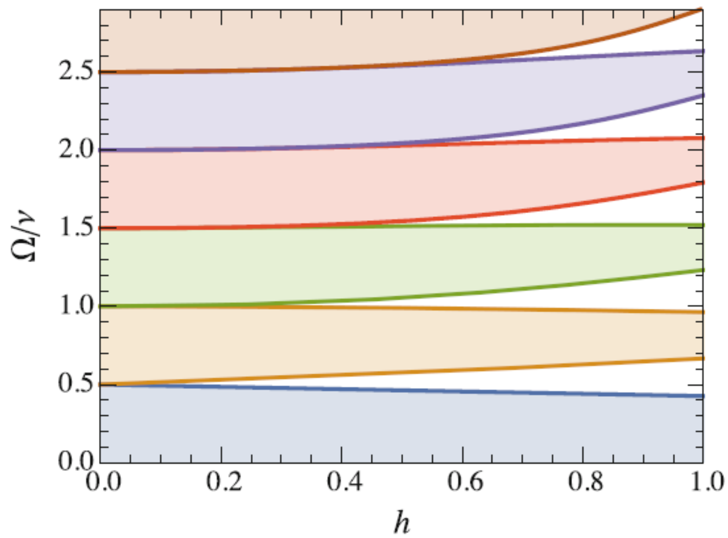
A well-known example is the Mathieu equation with the following time dependency.

$$\omega_0^2(t) = \Omega^2[1 - h\cos(\nu t)]. \tag{4.26}$$

- If h is small, (unless $\Omega/\nu = n/2$), solutions will be close to those of the harmonic oscillator with frequency equal to Ω.

- For small h, oscillations become unstable if $\Omega/\nu = n/2$.

$$\nu \approx \frac{2\Omega}{n} = 2\Omega, \Omega, \frac{2\Omega}{3}, \frac{2\Omega}{4}, \ldots \tag{4.27}$$



- The unstable gaps between the stable regions become exponentially narrow when $h \lesssim 1$ and Ω/ν increases.

- This means for a slow modulation $\nu \ll \Omega$, the region $h \lesssim 1$ can be considered as a practically stable area. This is the region of *adiabatically* slow variation of the oscillator parameters.

  – An adiabatically slow variation means

$$\omega_0^{-2} \left| \frac{d\omega_0}{dt} \right| \ll 1\,, \tag{4.28}$$

  which also means that the relative change of ω_0 over time ω_0^{-1} is small.

  – We seek a solution as a real part of the complex function $\xi(t)$:

$$\xi(t) = A(t) \exp\left(-i \int_0^t \omega_0(t')dt' + i\phi_0 \right), \tag{4.29}$$

  where $A(t)$ is the slowly varying amplitude and ϕ_0 is the initial phase.

  – Substituting into Eq. (4.25) yields

$$\ddot{A} - 2i\omega_0 \dot{A} - i\dot{\omega}_0 A = 0\,. \tag{4.30}$$

– Since we expect that A is a slow function of time, we neglect $\ddot{A}$.

$$2\omega_0 \dot{A} + \dot{\omega}_0 A = 0\,. \tag{4.31}$$

– Therefore,

$$\frac{d}{dt}\ln(A^2\omega_0) = 0\,, \tag{4.32}$$

or,

$$A(t)^2\omega_0(t) = \text{const.} \tag{4.33}$$

– We found an *adiabatic invariant*. $A(t)$ varies on the same time scale as $\omega_0(t)$, and hence is a slow-varying function as was assumed.

– The value of the constant is defined by the initial values of A and ω_0; 나중 시간에는 $A(t)^2\omega_0(t) = \text{const.} = A(0)^2\omega_0(0)$ 이기 때문에, $A \propto 1/\sqrt{\omega_0}$ 라고 볼 수 있다.

제 5 절 Nonlinear Oscillator and Nonlinear Resonance

- Accounting for higher order terms in the potential energy

$$\ddot{x} = -\omega_0^2 x + \alpha x^2 + \beta x^3 + \ldots\,, \tag{4.34}$$

where the terms on the right side of the equation are obtained through the Taylor expansion of the potential energy close to the equilibrium position.

- The oscillator is *weakly* nonlinear if the nonlinear terms with α and β are small in comparison with the linear ones.

- One of the most important properties of the nonlinear oscillator is that its frequency depends on the amplitude.

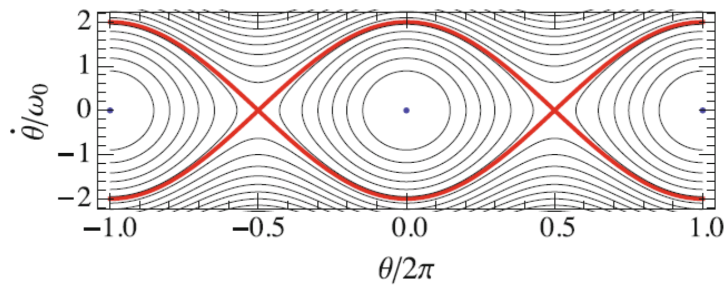
5.1 Example: Pendulum Equation

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0\,. \tag{4.35}$$

- For small amplitude, $|\theta| \ll 1$,

$$\sin\theta \approx \theta - \frac{1}{6}\theta^3\,, \tag{4.36}$$

이것은 $\alpha = 0$ 그리고 $\beta = \omega_0^2/6$ 에 해당.



- Using the integral of motion that characterizes the energy (1 장에서 배운 내용):

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega_0^2\cos\theta\right] = 0 \tag{4.37}$$

$$\dot{\theta} = \pm\omega_0\sqrt{2(E+\cos\theta)}\,. \tag{4.38}$$

  - *Phase portrait*: Plot of trajectories in the plane $(\theta, \dot{\theta}/\omega_0)$ using contours of constant E.
  - *Stable points*: $(2\pi n, 0)$ and $E = -1$.
  - *Unstable points*: $(2\pi n + \pi, 0)$ and $E = 1$.
  - *Separatrices*: Trajectories that pass through the unstable points are called separatrices. The separatrices are the orbits with energy $E = 1$.
  - *Bound motion* $(-1 < E < 1)$: These trajectories occupy a limited extension in θ.
  - *Unbound motion* $(E > 1)$: Pendulum rotates about the pivot point. The angle θ varies without limit.

- Period of pendulumn:

$$\frac{1}{2}T\omega_0 = \omega_0\int_{t_1}^{t_2} dt = \omega_0\int_{-\theta_0}^{\theta_0}\frac{d\theta}{\dot{\theta}} = \frac{1}{\sqrt{2}}\int_{-\theta_0}^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}\,. \tag{4.39}$$

  where θ_0 is defined by $E + \cos\theta_0 = 0$.

- In terms of the complete elliptic integral of the first kind K with $T_0 = 2\pi/\omega_0$:

$$\frac{T}{T_0} = \frac{2}{\pi}K\left[\sin^2\left(\frac{\theta_0}{2}\right)\right] = \frac{2}{\pi}K\left(\frac{1+E}{2}\right)\,, \tag{4.40}$$

  For small values of argument, $(2/\pi)K(x) \approx 1 + x/4$,

$$\omega \approx \omega_0\left(1 - \frac{\theta_0^2}{16}\right)\,. \tag{4.41}$$

  The frequency decreases with the amplitude θ_0.

5.2 Weakly nonlinear oscillator

- For $\omega_0 \gg |\nu|a^2$, we expect a nonlinear correction to the frequency ω_0,

$$\omega(a) \approx \omega_0 + \nu a^2 \,, \tag{4.42}$$

  where a is the amplitude and ν is a constant. Detailed calculations show

$$\nu = -\frac{3\beta}{8\omega_0} - \frac{5\alpha^2}{12\omega_0^3} \,. \tag{4.43}$$

- *Anharmonicity*: Fourier spectrum of a weakly nonlinear oscillation contains not only the fundamental frequency $\omega(a)$, but also small contributions from higher harmonics $n\omega(a)$.

5.3 Nonlinearity also changes the resonance effect

For a linear oscillator an external force at the resonant frequency can drive the oscillator amplitude to very large values, if the damping is small. The situation is different for a nonlinear oscillator for a reason that is easy to understand: when the amplitude grows, the frequency of the oscillator drifts from its initial value, detuning the oscillator from the resonance.

- The unlimited growth of the amplitude ceases when the amplitude reaches some value a_* which depends on the strength of the external force and the nonlinearity. In Eq. (4.17),

$$|\xi_0|^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \,. \tag{4.44}$$

  we set $\gamma \to 0$, $\omega_0 \to \omega_0 + \nu a_*^2$, and $\omega = \omega_{\text{driving}} \to \omega_0$. For small νa_*^2, we find

$$a_*^2 \approx \frac{f_0^2}{(2|\nu|\omega_0 a_*^2)^2} \,. \tag{4.45}$$

  Solving for a_* yields

$$a_* \approx \left(\frac{f_0}{2|\nu|\omega_0}\right)^{1/3} \,. \tag{4.46}$$

- Due to the nonlinearity, even at exact resonance the amplitude of the oscillations is finite

- For drive frequencies different from ω_0, this dependence will change and the peak amplitude may even increase for a forcing term having the same magnitude.