제 3 장

Action-angle variables and Liouville's theorem

Canonical transformation이 사용되는 가장 대표적인 예는 동력학 문제를 action과 angle이라는 두 개의 변수로 표현하는 것이다. 이 경우 Hamiltonian 역학을 매우 단순하게 취급할 수 있도록 해주기 때문에, 가속기물리에서 매우 유용하다. 이 장 후반부에서는 Liouville's theorem을 Hamiltonian 역학의 관점에서 공부한다. Liouville's theorem is crucial for understanding the fundamental properties of large ensembles of beam particles in accelerators.

제 1 절 Canonical Transformation for a Linear Oscillator

An illustrative example of the canonical transformation: a simple harmonic oscillator

- The Hamiltonian for an oscillator with a unit mass:

$$H(x,p) = \frac{p^2}{2} + \frac{\omega_0^2 x^2}{2}\,, \tag{3.1}$$

- The equations of motion:

$$\dot{p} = -\frac{\partial H}{\partial x} = \omega_0^2 x\,, \qquad \dot{x} = \frac{\partial H}{\partial p} = p\,, \tag{3.2}$$

- Solution:

$$x = a\cos(\omega_0 t + \phi_0)\,, \qquad p = -a\omega_0 \sin(\omega_0 t + \phi_0)\,, \tag{3.3}$$

- We would like to find a canonical transformation from the old variables, x, p, to the new ones, ϕ, I, (where ϕ is the new coordinate and I is the new momentum):

$$x = A(I)\cos\phi\,, \qquad p = -A(I)\,\omega_0 \sin\phi\,, \tag{3.4}$$

식 (3.3)과 비교를 하면, $a \to A(I)$에 대응되므로, I는 constant of motion이고, $\omega_0 t + \phi_0 \to \phi$이므로, ϕ는 시간에 대한 선형 함수이다.

- To construct the canonical transformation (3.4) we will use the generating function $F_1(x, \phi)$ of the first type.

- First, we express p in terms of the old (x) and new (ϕ) coordinates by eliminating $A(I)$ from (3.4):

$$p = -\omega_0 x \tan\phi\,. \tag{3.5}$$

- 2장에 소개된 F_1에 대한 미분식을 적분하면, F_1 을 얻을 수 있다. 여기서는 적분 상수는 없어도 됨.

$$\left(\frac{\partial F_1}{\partial x}\right)_\phi = p = -\omega_0 x \tan\phi$$

$$F_1(x, \phi) = \int p\,dx = -\frac{\omega_0 x^2}{2}\tan\phi\,. \tag{3.6}$$

- F_1을 다시 새로운 좌표계에 대해 미문하면, 새로운 momentum을 얻을 수 있다.

$$I = -\frac{\partial F_1}{\partial \phi} = \frac{\omega_0 x^2}{2}(1 + \tan^2\phi) = \frac{1}{2\omega_0}\left(\omega_0^2 x^2 + p^2\right)\,, \tag{3.7}$$

- By substituting Eq. (3.4) into (3.7):

$$A(I) = \sqrt{\frac{2I}{\omega_0}}. \tag{3.8}$$

- The new coordinate in terms of old variables:

$$\phi = -\arctan\frac{p}{\omega_0 x}\,. \tag{3.9}$$

- Because the canonical transformation does not depend on time, the new Hamiltonian is equal to the old one expressed in new variables.

$$H' = H + \frac{\partial F_1}{\partial t} = H$$

- Comparing Eq. (3.1) with (3.7):

$$H' = \omega_0 I\,. \tag{3.10}$$

I 의 단위는 ? [에너지] $\times$ [시간]

- The equations of motion in new variables

$$\dot{I} = -\frac{\partial H'}{\partial \phi} = 0\,, \qquad \dot{\phi} = \frac{\partial H'}{\partial I} = \omega_0\,. \tag{3.11}$$

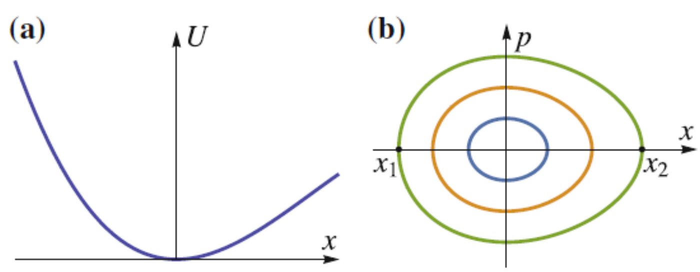
$$I = \text{const}\,, \qquad \phi = \omega_0 t + \phi_0\,. \tag{3.12}$$

  Of course, this is the same dynamics as described by the original Eq. (3.3), but it is simpler because one of the coordinates, I, turns out to be an integral of motion and the other one, ϕ, is a simple linear function of time.

- The (I, ϕ) pair is called the *action-angle* coordinates for this particular case. They are especially useful for building perturbation theory for more complicated systems that in the lowest approximation reduce to a linear oscillator.

제 2 절 Action-Angle Variables in 1D

We can generalize the action-angle variables introduced in the previous section to 1D periodic (x가 에너지에 따라 주어진 구간을 반복해서 왕복운동) motion in an *arbitrary* but constant (시간에 대해 의존하지 않는다는 이야기) potential well $U(x)$. $U(x) \propto x^2$ 인 특별한 경우가 simple harmonic oscillator 에 해당.



2.1 Energy E 를 새로운 모멘텀으로 한 경우

- Hamiltonian for this problem (assuming a unit mass):

$$H(x,p) = \frac{p^2}{2} + U(x)\,. \tag{3.13}$$

- Each trajectory in phase space is defined by a constant value of the Hamiltonian, $H(x, p) = E$, where E is the energy.

- Both x and p for a given trajectory are periodic functions of time oscillating with the revolution frequency ω that depends on the energy, $\omega(E)$.

$$\frac{dx}{dt} = p = \sqrt{2(E - U(x))} \quad (\because m = 1) \tag{3.14}$$

$$\frac{1}{2}T = \pi\omega^{-1} = \int_{x_1}^{x_2} \frac{dx'}{p(x')} = \int_{x_1}^{x_2} \frac{dx'}{\sqrt{2(E-U(x'))}}\,, \tag{3.15}$$

where $T = 2\pi/\omega$ is the period, and x_1 and x_2 are the turning points on the orbit.

- Canonical transformation from (x, p) to (Q, E): $F_2(x, E)$

$$\frac{\partial F_2}{\partial x} = p = \sqrt{2(E - U(x))} \tag{3.16}$$

$$F_2(x, E) = \int^x dx' \sqrt{2(E - U(x'))}\,. \tag{3.17}$$

- This is a time-independent transformation.

$$H'(Q, E) = H + \cancel{\frac{\partial F_2}{\partial t}} = E\,, \tag{3.18}$$

- Equations of motion for the new variables:

$$\dot{Q} = \frac{\partial H'}{\partial E} = 1\,, \qquad \dot{E} = -\frac{\partial H'}{\partial Q} = 0\,. \tag{3.19}$$

- The evolution of the variable Q is simple.

$$Q = t + t_0\,. \tag{3.20}$$

- 문제점: 한 바퀴를 돌고 나면, Q는 한 주기 $T = 2\pi/\omega(E)$ 만큼 증가하게 되는데, 이 주기가 E 의 함수가 되어버려서, 에너지 E에 따라 각각의 궤적의 주기가 달라지게 된다.

2.2 Action J 를 새로운 모멘텀으로 한 경우

좀더 좋은 방법은 새로운 좌표계로 각도 (angle) ϕ 를 잡아서, 한 주기 후에는 모든 궤적이 2π 만큼 공통적으로 움직이도록 하는 것이다.

- The new coordinate ϕ is called the *angle*, and the corresponding generalized momentum, J, is the *action.*

- Canonical transformation from (x, p) to (ϕ, J): $\tilde{F}_2(x, J)$

- The action is a function of energy, $J(E)$, or, conversely, $E = E(J)$.

- The generating function is only slightly modified.

$$\tilde{F}_2(x, J) = F_2(x, E(J))\,. \tag{3.21}$$

- As the Hamiltonian is time-independent, the new Hamiltonian is

$$\tilde{H}(\phi, J) = E(J)\,, \tag{3.22}$$

- The equations of motion for the new variables:

$$\dot{\phi} = \frac{\partial \tilde{H}}{\partial J} = \frac{dE}{dJ} = \omega(E), \quad \dot{J} = -\frac{\partial \tilde{H}}{\partial \phi} = 0\,. \tag{3.23}$$

- Integrating the equation with $\dot{\phi} = \omega(E)$ gives

$$\phi = \omega(E)t + \phi_0\,. \tag{3.24}$$

  With this time dependence, one orbital period corresponds to the change of ϕ by 2π, as desired. 정의에 의해 $\omega(E)T = 2\pi$ 이다.

- Integrating the differential equation for $E(J)$

$$\frac{dE}{dJ} = \omega(E)\,. \tag{3.25}$$

$$J(E) = \int_{E_{\min}}^{E} \frac{dE'}{\omega(E')}\,, \tag{3.26}$$

  where $E_{\min}$ is the energy corresponding to the bottom of the potential well U.

NOTES

- Generating function 이 꼭 해석해로 존재할 필요는 없다. 수치해로 존재해도 상관없음.
- The key features of action-angle coordinates are that the action is a constant of the motion, and the angle grows linearly in time, with periodic motion corresponding to a change in phase of 2π.
- The rate of change in the phase ($\dot{\phi} = \omega(E)$) is generally different for different trajectories. The simple harmonic oscillator is a notable exception where all trajectories have the same period ($\dot{\phi} = \omega_0$).

제 3 절 Hamiltonian Flow in Phase Space and Symplectic Maps

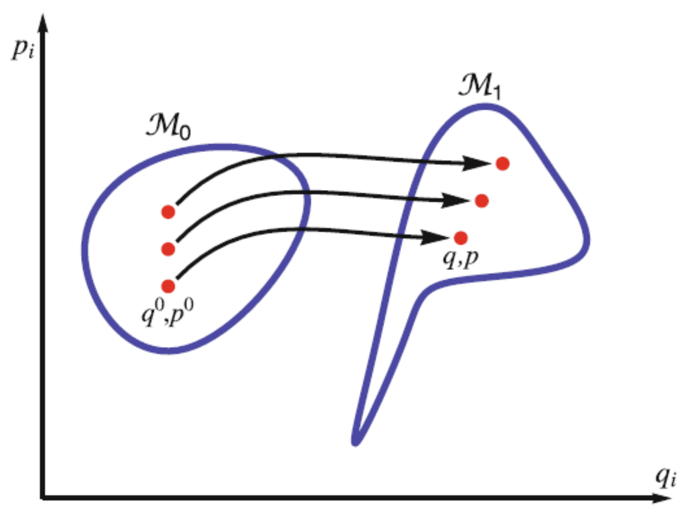
3.1 Hamiltonian 운동을 기하학적인 관점으로 보기

- A *map* of the initial domain in the $2n$ dimensional phase space to a manifold (다양체) in the same phase space at time t:

$$q_i = q_i(q_k^0, p_k^0, t_0, t)\,, \qquad p_i = p_i(q_k^0, p_k^0, t_0, t)\,. \tag{3.27}$$

  Varying t in these equations moves each point (q_i, p_i) along its trajectory and the set of all trajectories starting from the initial domain constitutes a *Hamiltonian flow*.

- For a given t_0 and t, Eq. (3.27) constitute a canonical transformation from q_i^0, p_i^0 to q_i, p_i, which is also called a *symplectic transfer map*.



3.2 $n = 1$ 인 경우에 대하여, symplectic transfer map 이 canonical transformation 이 됨을 증명

- 먼저, $t = t_0$에 대하여 아래의 사실은 자명함.

$$\{q,p\}_{q^0,p^0} = 1, \quad \{p,p\}_{q^0,p^0} = \{q,q\}_{q^0,p^0} = 0 \tag{3.28}$$

- $\{p,p\}_{q^0,p^0} = \{q,q\}_{q^0,p^0} = 0$ 은 시간에 관계 없이 항상 자명하게 성립.

- 따라서, 위의 Poisson bracket 중 $\{q,p\}_{q^0,p^0}$ 이 시간에 대해 변하지 않는다는 것만 보이면, 증명이 끝남.

$$\begin{aligned}\frac{d}{dt}\{q,p\}_{q^0,p^0} &= \frac{d}{dt}\left(\frac{\partial q}{\partial q^0}\frac{\partial p}{\partial p^0} - \frac{\partial q}{\partial p^0}\frac{\partial p}{\partial q^0}\right) \\ &= \frac{\partial p}{\partial p^0}\frac{\partial}{\partial q^0}\frac{dq}{dt} + \frac{\partial q}{\partial q^0}\frac{\partial}{\partial p^0}\frac{dp}{dt} - \frac{\partial p}{\partial q^0}\frac{\partial}{\partial p^0}\frac{dq}{dt} - \frac{\partial q}{\partial p^0}\frac{\partial}{\partial q^0}\frac{dp}{dt} \\ &= \frac{\partial p}{\partial p^0}\frac{\partial}{\partial q^0}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial q^0}\frac{\partial}{\partial p^0}\frac{\partial H}{\partial q} - \frac{\partial p}{\partial q^0}\frac{\partial}{\partial p^0}\frac{\partial H}{\partial p} + \frac{\partial q}{\partial p^0}\frac{\partial}{\partial q^0}\frac{\partial H}{\partial q}\,.\end{aligned} \tag{3.29}$$

Applying the chain rules,

$$\frac{\partial}{\partial p^0} = \frac{\partial p}{\partial p^0}\frac{\partial}{\partial p} + \frac{\partial q}{\partial p^0}\frac{\partial}{\partial q}, \qquad \frac{\partial}{\partial q^0} = \frac{\partial p}{\partial q^0}\frac{\partial}{\partial p} + \frac{\partial q}{\partial q^0}\frac{\partial}{\partial q}, \tag{3.30}$$

we obtain

$$\frac{d}{dt}\{q,p\}_{q^0,p^0} = 0 \tag{3.31}$$

보충: Interchange property of partial and ordinary derivatives

Let $f = f(q,p,t)$. The ordinary derivative of f with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial t} \tag{3.32}$$

$$\frac{\partial}{\partial q}\left(\frac{df}{dt}\right) = \frac{\partial^2 f}{\partial q\partial q}\dot{q} + \frac{\partial^2 f}{\partial q\partial p}\dot{p} + \frac{\partial^2 f}{\partial q\partial t} \tag{3.33}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial q}\right) = \frac{\partial^2 f}{\partial q\partial q}\dot{q} + \frac{\partial^2 f}{\partial p\partial q}\dot{p} + \frac{\partial^2 f}{\partial t\partial q} \tag{3.34}$$

Therefore,

$$\frac{\partial}{\partial q}\left(\frac{df}{dt}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial q}\right) \tag{3.35}$$

여기서, $\dot{q}$ 및 $\dot{p}$ 는 q 에 독립적인 변수로 간주.

3.3 Symplectic maps

Canonical transformation Eq. (3.27) 을 종종 symplectic (transfer) map 으로 기술하기 위해 아래의 block-diagonal $2n \times 2n$ antisymmetric matrix 를 도입.

$$J_{2n} = \begin{pmatrix} J_2 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_2 \end{pmatrix}, \tag{3.36}$$

The diagonal elements are

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.37}$$

주의: Wolski 책에서는 $J_2 \to -J_2 = S_2$로 정의됨.

- Notation for $2n$ variables consistent with the definition of J_{2n}:

$$w_k: \quad w_{2k-1} = q_k, \quad w_{2k} = p_k, \quad k = 1, 2, \ldots, n \tag{3.38}$$

$$W_i: \quad W_{2i-1} = Q_i, \quad W_{2i} = P_i, \quad i = 1, 2, \ldots, n \tag{3.39}$$

- The transformation from the old to new variables (2.2) is then replaced by $2n$ functions as

$$W_i = W_i(w_k, t), \qquad i, k = 1, 2, \ldots, 2n\,. \tag{3.40}$$

즉, 위의 식은 아래와 등가적임.

$$Q_i = Q_i(q_k, p_k, t)\,, \qquad P_i = P_i(q_k, p_k, t)\,, \qquad i = 1, \ldots, n \tag{3.41}$$

- The requirement that all possible Poisson brackets satisfy Eqs. (2.15)–(2.17) (which, as we know, is equivalent to the requirement for the transformation to be canonical) can be concisely written as

$$M J_{2n} M^T = J_{2n}, \text{ or } \; M^T J_{2n} M = J_{2n} \tag{3.42}$$

where M is the Jacobian matrix of the transformation

$$M_{ij} = \frac{\partial W_i}{\partial w_j} \tag{3.43}$$

A transformation with Jacobian satisfying Eq. (3.42) is said to be *sympletic*. The Jacobian of a canonical transformation is a symplectic matrix.

- Both the matrix and its determinant are referred to as the Jacobian in some literatures.

보충: Proof that Jacobian matrix of a canonical transformation is symplectic (A. Wolski 책 내용)

Hamiltonian equations with old and new variables:

$$\dot{w}_i = S_{ik} \frac{\partial H}{\partial w_k}, \qquad \dot{W}_i = S_{ik} \frac{\partial H}{\partial W_k}, \qquad i, k = 1, 2, \ldots, 2n \tag{3.44}$$

Here,

$$S = \begin{pmatrix} S_2 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & S_2 \end{pmatrix} = -J_{2n}\,,$$

Consider an infinitesimal change in the independent variable from t_0 to $t_0 + \delta t$:

$$W_i(t_0+\delta t) = W_i(t_0) + \dot{W}_i(t_0)\delta t = w_i(t_0) + \dot{w}_i(t_0)\delta t = w_i + S_{ik}\left(\frac{\partial H}{\partial w_k}\right)_{t_0} \delta t \qquad (3.45)$$

From the definition of the Jacobian of the transformation from t_0 to $t_0 + \delta t$ (we note that $S_{ii} = 0$)

$$M_{ij} = \frac{\partial W_i(t_0+\delta t)}{\partial w_j} = \delta_{ij} + S_{ik}\left(\frac{\partial^2 H}{\partial w_j \partial w_k}\right)_{t_0} \delta t = \delta_{ij} + S_{ik}\tilde{H}_{kj}\delta t \qquad (3.46)$$

Or, in the matrix notation

$$M = I + S\tilde{H}\delta t \qquad (3.47)$$

To first order in δt

$$(I + S\tilde{H}\delta t)S(I + S\tilde{H}\delta t)^T = (I + S\tilde{H}\delta t)S(I - \tilde{H}S\delta t) = S \qquad (3.48)$$

Or, equivalently,

$$(I + S\tilde{H}\delta t)^T S(I + S\tilde{H}\delta t) = (I - \tilde{H}S\delta t)S(I + S\tilde{H}\delta t) = S \qquad (3.49)$$

If we think of expansion of W_i about the reference orbit:

$$W_i(w_j, t_0+\delta t) = W_i(0, t_0+\delta t) + \sum_j \frac{\partial W_i(0, t_0+\delta t)}{\partial w_j} w_j = \sum_j M_{ij}(0) w_j \qquad (3.50)$$

Here, we assume that the reference orbit is transformed to a reference orbit when there is no constant force term (i.e, linear term in the Hamiltonian), and set $W_i(0, t_0+\delta t) = 0$.

NOTES

1. 만약, H 가 w_i 에 대한 2차식으로만 이루어져 있다면 (즉, 운동방정식이 선형이라면), $M_{ij}(0) = M_{ij}$이고 mapping 을 다음과 같이 행렬 형태로 표현할 수 있다.

$$W_i(t_0+\delta t) = M_{ij} w_j \qquad (3.51)$$

2. 만약, H 가 w_i 에 대한 1, 2차식 모두로 이루어져 있다면 0th order 항이 생긴다.

$$W_i(t_0+\delta t) = M_{ij} w_j + m_i \qquad (3.52)$$

3. 만약, 예 1에서 H 가 시간에 대한 의존성이 없고 $t = t_0 + N\delta t = t_0 + \Delta t$ 에서 계산하면

$$W_i(t_0 + \Delta t) = [I + S\tilde{H}\delta t]^N_{ij} w_j \simeq [I + S\tilde{H}\Delta t]_{ij} w_j \tag{3.53}$$

4. 만약, 예 1에서 H 가 piece-wise constant 하다면, 즉, Δt_1 동안에 H_1 을 가지고, Δt_2 동안에 H_2 을 가지고, 등등

$$W_i(t_0 + \Delta t_1 + \Delta t_2 + \cdots) = \Big[(\cdots)(I + S\tilde{H}_2\Delta t_2)(I + S\tilde{H}_1\Delta t_1)\Big]_{ij} w_j \tag{3.54}$$

5. 기본적으로 예 3, 4 에서는 Δt 등이 모두 매우 작을 때 이며, 그렇지 않을 경우 higher-order 항들을 모두 고려해야 한다. 예를 들면 quadrupole 의 transfer matrix 는 아래과 같이 주어진다. 이 식은 기본적으로 운동방정식을 직접 풀어서 $W_i = M_{ij} w_j$ 로 표현한 후, $M_{ij} = \partial W_i / \partial w_j$ 를 계산한 것이다. 즉, 모든 higher-order 항들 고려한 것이된다.

$$M_{ij} = \begin{pmatrix} \cos(\sqrt{k}l) & \frac{\sin(\sqrt{k}l)}{\sqrt{k}} \\ -\sqrt{k}\sin(\sqrt{k}l) & \cos(\sqrt{k}l) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & l \\ -kl & 1 \end{pmatrix} \tag{3.55}$$

우변의 근사는 l (또는 이 논의에서는 Δt 에 해당) 을 매우 작다고 가정하여 얻은 것이며, 이 경우 Hamiltonian 으로 부터 Jacobian matrix 를 계산하여서도 얻을 수 있다.

$$H = \frac{p_x^2}{2} + \frac{k}{2}x^2 \tag{3.56}$$

$$\tilde{H}_{kj} = \left(\frac{\partial^2 H}{\partial w_j \partial w_k}\right)_{t_0} \longrightarrow \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \tag{3.57}$$

$$M_{ij} = \delta_{ij} + S_{ik}\tilde{H}_{kj} l \longrightarrow \begin{pmatrix} 1 & l \\ -kl & 1 \end{pmatrix} \tag{3.58}$$

제 4 절 Liouville's Theorem

- Volume of the old phase space domain:

$$V_1 = \int_{\mathcal{M}_0} dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n \,, \tag{3.59}$$

- Volume of the new phase space domain:

$$V_2 = \int_{\mathcal{M}_1} dQ_1 dQ_2 \ldots dQ_n dP_1 dP_2 \ldots dP_n \,. \tag{3.60}$$

- It turns out that $V_2 = V_1$.

- Proof: The ratio of infinitesimal volumes in a transformation of variables is equal to the absolute value of the determinant of the Jacobian matrix of the transformation.

$$\frac{dQ_1 dQ_2 \ldots dQ_n dP_1 dP_2 \ldots dP_n}{dq_1 dq_2 \ldots dq_n dp_1 dp_2 \ldots dp_n} = |\det M| \,. \tag{3.61}$$

Since,

$$\det(MJ_{2n}M^T) = (\det M)^2 \det(J_{2n}) = \det(J_{2n}) = 1 \tag{3.62}$$

Therefore, $|\det(M)| = 1$.

- Liouville's theorem guarantees that the phase space volume occupied initially by a beam remains the same through its Hamiltonian evolution with time.

보충: Distribution function

- It is a continuous (mathematical) approximation of discrete (real) particle distribution.

- The number of particles found in a differential volume in the neighborhood of a phase space location x, p at a time t:

$$f(\mathbf{x}, \mathbf{p}, t) d^3\mathbf{x} d^3\mathbf{p} \tag{3.63}$$

- With a smooth phase space distribution, the charge and current distributions associated with such a distribution are also continuous and smooth.

- The fields derived from the smooth charge/current densities may be termed macroscopic. Deviations from these approximate fields (near an individual particle) may be termed microscopic.

보충: Proof of Liouville's theorem

- Total time derivative of the distribution function:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \nabla f + \dot{\mathbf{p}} \cdot \nabla_{\mathbf{p}} f \tag{3.64}$$

- From continuity in phase-space:

$$0 = \frac{\partial f}{\partial t} + \nabla \cdot (\dot{\mathbf{x}} f) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}} f) \tag{3.65}$$

- If the forces are derivable from a Hamiltonian, then

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_i \left(\frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dp_i}{dt} \frac{\partial f}{\partial p_i} \right) \\ &= -\sum_i \left(\frac{\partial \dot{x}_i}{\partial x_i} f + \frac{\partial \dot{p}_i}{\partial p_i} f \right) = -\sum_i f \left[\frac{\partial}{\partial x_i} \left(\frac{\partial H}{\partial p_i} \right) - \frac{\partial}{\partial p_i} \left(\frac{\partial H}{\partial x_i} \right) \right] = 0 \end{aligned}$$

- In other words, when no dissipative forces, no particle lost or created, and no small-impact-parameter binary Coulomb collisions between particles, we have

$$\frac{df}{dt} = 0 \tag{3.66}$$

- Incompressibility:

$$\nabla \cdot (\dot{\mathbf{x}}) + \nabla_{\mathbf{p}} \cdot (\dot{\mathbf{p}}) = 0 \tag{3.67}$$

보충: Comments on Liouville's theorem

- Liouville's theorem states that the phase space density encountered as one travels with a particle in a Hamiltonian system is conserved.
  - The density of any volume of phase space whose boundary follows the Hamiltonian equations is constant.
  - The volume occupied by particles in phase space ($\sim$emittance) is conserved (shape may change).
- Liouville's theorem is valid not only for the time-independent Hamiltonian case, but also for the time-dependent Hamiltonian case.
- Liouville's theorem is valid for both equilibrium and non-equilibrium systems.
- Liouville's theorem is valid for both linear and non-linear systems.
- Liouville's theorem does not imply that the density is uniform throughout phase space.
- Liouville's theorem only holds in the limit that the particles are infinitely close together. Equivalently, Liouville's theorem does not hold for any ensemble that consists of a finite number of particles.
- Liouville's theorem holds even in the presence of space-charge and wake-fields, but not with microscopic binary collisions.

보충: Meaning of equilibrium

- Any positive-definite (because it should represent particle counts) distribution function formed from a set of single-particle constants of the motion $\{C_i\}$ will produce a valid, exact 'equilibrium' solution to the Vlasov equation.

$$\frac{d}{dt} f(\{C_i\}) = 0 \tag{3.68}$$

- A special case is a stationary (time-independent) equilibrium with $\partial/\partial t = 0$. Stationary beam equilibria occur in continuous-focusing systems.

- In continuous-focusing systems, one may assume the beam is in thermal equilibrium

$$f \propto \exp\left[-\frac{H}{k_B T_{eq}}\right], \quad \frac{\partial f}{\partial t} = 0, \tag{3.69}$$

  where H is a constant of the motion for a time-independent Hamiltonian,

- In non-continuous lattices, projections of the distribution can evolve in t .

- In the periodic focusing system, the particle distribution is non-stationary, however, when plotted in trace space once per period (i.e., in the Poincare plot), we can treat the beam in stationary equilibrium.

$$f(t) = f(t+T) \tag{3.70}$$

제 5 절 Non-conservative Forces in Hamiltonian Dynamics

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