제 2 장

Canonical transformations

새로운 generalized coordinates Q_i를 정의하면, Lagrangian으로 부터 새로운 momenta P_i를 유도할 수 있다. 이경우 Hamiltonian도 $H'(Q_i, P_i, t)$로 새롭게 정의 된다. 이러한 Q_i 및 P_i를 잘 선택하면, 우리가 다루는 동역학 시스템을 매우 쉽게 단순화 할 수 있다. Let us assume that have a set of canonical variables q_i, p_i and the corresponding Hamiltonian $H(q_i, p_i, t)$, and then make a transformation to new variables

$$Q_i = Q_i(q_k, p_k, t)\,, \qquad P_i = P_i(q_k, p_k, t)\,, \qquad i = 1 \ldots n. \tag{2.1}$$

Can we find a new Hamiltonian $H'(Q_i, P_i, t)$ such that the dynamics as expressed in the new variables is also Hamiltonian? What are the requirements on the transformation (2.1) for such a Hamiltonian to exist? 이러한 질문에 대한 답을 체계화 한 것이 'Canonical Transformation'이다.

제 1 절 Canonical Transformations

- We first consider a *time-independent* Hamiltonian $H(q_i, p_i)$, and later generalize the result for the case when H is a function of time.

$$Q_i = Q_i(q_k, p_k)\,, \qquad P_i = P_i(q_k, p_k)\,, \qquad i = 1 \ldots n, \tag{2.2}$$

- An inverse transformation:

$$q_i = q_i(Q_k, P_k)\,, \qquad p_i = p_i(Q_k, P_k)\,, \qquad i = 1 \ldots n. \tag{2.3}$$

  It is obtained by considering Eq. (2.2) as $2n$ equations for the old variables and solving them for q_i, p_i.

- Hamiltonian in terms of the new variables:

$$H'(Q_k, P_k) = H(q_i(Q_k, P_k), p_i(Q_k, P_k)), \tag{2.4}$$

  2.4 절에 설명되겠지만, 반드시 이식이 만족될 필요는 없다. (즉, 일반적으로 $H' \neq H$)

- Let us assume that we have solved the Hamiltonian equations of motion and found a trajectory $(q_i(t), p_i(t))$.

$$Q_i(t) = Q_i(q_k(t), p_k(t))\,, \qquad P_i(t) = P_i(q_k(t), p_k(t))\,. \tag{2.5}$$

- We would like the new trajectory to be a Hamiltonian orbit; it has to satisfy the equations

$$\frac{dP_i}{dt} = -\frac{\partial H'(Q_k(t), P_k(t))}{\partial Q_i}\,, \qquad \frac{dQ_i}{dt} = \frac{\partial H'(Q_k(t), P_k(t))}{\partial P_i}\,. \tag{2.6}$$

If the transformation (2.2) is such that Eq. (2.6) is satisfied for every Hamiltonian H, then it is called a *canonical transformation*.

- Example:

$$Q_i = p_i\,, \qquad P_i = -q_i\,. \tag{2.7}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\,, \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \;\rightarrow\; \frac{dp_i}{dt} = \frac{\partial H}{\partial(-q_i)}\,, \frac{d(-q_i)}{dt} = -\frac{\partial H}{\partial p_i}$$

or

$$Q_i = -p_i\,, \qquad P_i = q_i\,. \tag{2.8}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\,, \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \;\rightarrow\; \frac{d(-p_i)}{dt} = \frac{\partial H}{\partial q_i}\,, \frac{dq_i}{dt} = -\frac{\partial H}{\partial(-p_i)}$$

이 예에서는 모멘텀과 좌표의 부호를 적절히 바꾸어 그 역할을 서로 바꾸는 것을 보여준다. 즉, 헤밀토니안 역학에서는 conjugate variables는 서로 동등한 역할을 한다. (라그랑지안 역학에서는 좌표와 속도의 역할이 서로 다름.)

제 2 절 Poisson Brackets and Canonical Transformations

How to test that a given transformation is canonical:

- For two functions of canonical variables, $f(q_i, p_i)$ and $g(q_i, p_i)$:

$$\{f, g\}_{q,p} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \equiv J(q, p)\,, \tag{2.9}$$

- Using the inverse transformation (2.3) we can express our functions in terms of the new variables Q_i and P_i; the resulting new functions are denoted by $f'(Q_i, P_i)$ and $g'(Q_i, P_i)$.

$$\{f', g'\}_{Q,P} = \sum_i \left(\frac{\partial f'}{\partial Q_i}\frac{\partial g'}{\partial P_i} - \frac{\partial f'}{\partial P_i}\frac{\partial g'}{\partial Q_i} \right) \equiv J'(Q, P)\,. \tag{2.10}$$

- If $(q_i, p_i) \to (Q_i, P_i)$ is a canonical transformation, then rewriting $J'(Q, P)$ by expressing the new variables in terms of the old ones gives $J(q, p)$:

$$J'(Q_i(q_k, p_k), P_i(q_k, p_k)) = J(q_k, p_k)\,. \tag{2.11}$$

or

$$\{f, g\}_{q,p} = \{f', g'\}_{Q,P}\,, \tag{2.12}$$

In practice, to establish that a transformation is canonical, we do not need to verify the equality (2.12) for all possible functions f and g; it is enough to make sure that it holds for a special set of $3n^2$ pairs of functions.

- For $f = Q_i(q_l, p_l)$ and $g = Q_k(q_l, p_l)$:

$$\{Q_i, Q_k\}_{q,p} = \{q_i, q_k\}_{q,p} = 0\,. \tag{2.13}$$

- For $f = P_i(q_l, p_l)$ and $g = P_k(q_l, p_l)$:

$$\{P_i, P_k\}_{q,p} = \{p_i, p_k\}_{q,p} = 0\,. \tag{2.14}$$

- For $f = Q_i(q_l, p_l)$ and $g = P_k(q_l, p_l)$:

$$\{Q_i, P_k\}_{q,p} = \{q_i, p_k\}_{q,p} = \delta_{ik}\,. \tag{2.15}$$

제 3 절 Generating Functions

Poisson brackets are useful for establishing that a given transformation is canonical. They do not, however, provide a tool with which one can create canonical transformations. A technique that allows one to do that is based on the approach that uses so-called *generating functions*.

- Example: A time-independent generating function of first type, $F_1(q_i, Q_i)$ which depends on $2n$ variables.

$$F_1(q_i, Q_i)\,. \tag{2.16}$$

$$p_k = \frac{\partial F_1(q_i, Q_i)}{\partial q_k}\,, \qquad P_k = -\frac{\partial F_1(q_i, Q_i)}{\partial Q_k}\,, \qquad k = 1 \ldots n\,. \tag{2.17}$$

- For $n = 1$:

$$Q = Q(q, p)\,, \qquad P = P(q, p)\,. \tag{2.18}$$

$$p = \frac{\partial F_1(q, Q)}{\partial q}\,, \qquad P = -\frac{\partial F_1(q, Q)}{\partial Q}\,. \tag{2.19}$$

Since, trivially

$$\{Q, Q\} = \{P, P\} = 0,$$

we only need to prove that

$$\{Q, P\}_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1\,. \tag{2.20}$$

- Proof:

$$\frac{\partial P}{\partial q} = -\frac{\partial^2 F_1}{\partial q \partial Q} - \frac{\partial^2 F_1}{\partial Q^2}\frac{\partial Q}{\partial q}\,, \qquad \frac{\partial P}{\partial p} = -\frac{\partial^2 F_1}{\partial Q^2}\frac{\partial Q}{\partial p}\,. \tag{2.21}$$

$$\{Q, P\}_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{\partial Q}{\partial p}\frac{\partial^2 F_1}{\partial q \partial Q}\,. \tag{2.22}$$

From

$$p = \frac{\partial F_1(q, Q)}{\partial q} \longrightarrow \frac{\partial p}{\partial p} = 1 = \frac{\partial}{\partial p}\left(\frac{\partial F_1(q, Q)}{\partial q}\right)$$

$$1 = \frac{\partial^2 F_1}{\partial Q \partial q}\frac{\partial Q}{\partial p}\,. \tag{2.23}$$

Finally, we obtain the desired relation:

$$\{Q, P\}_{q,p} = 1\,. \tag{2.24}$$

제 4 절 Transformations with a Time Dependence and Four Types of Generating Functions

A time-dependent canonical transformation adds the time variable to the relations between the old and new variables:

$$Q_i = Q_i(q_k, p_k, t)\,, \qquad P_i = P_i(q_k, p_k, t)\,, \qquad i = 1 \ldots n. \tag{2.25}$$

Furthermore, we will no longer insist on the particular relation $H = H'$ between the old and the new Hamiltonians, and allow for a broader class of functions $H'(Q_k, P_k, t)$.

- First type:

$$p_i = \frac{\partial F_1}{\partial q_i}\,, \qquad P_i = -\frac{\partial F_1}{\partial Q_i}\,, \tag{2.26}$$

$$H' = H + \frac{\partial F_1}{\partial t}\,. \tag{2.27}$$

- Second type:

$$p_i = \frac{\partial F_2}{\partial q_i}\,, \qquad Q_i = \frac{\partial F_2}{\partial P_i}\,, \tag{2.28}$$

$$H' = H + \frac{\partial F_2}{\partial t}\,. \tag{2.29}$$

- Third type:

$$q_i = -\frac{\partial F_3}{\partial p_i}\,, \qquad P_i = -\frac{\partial F_3}{\partial Q_i}\,, \tag{2.30}$$

$$H' = H + \frac{\partial F_3}{\partial t}\,, \tag{2.31}$$

- Fourth type:

$$q_i = -\frac{\partial F_4}{\partial p_i}\,, \qquad Q_i = \frac{\partial F_4}{\partial P_i}\,, \tag{2.32}$$

$$H' = H + \frac{\partial F_4}{\partial t}\,. \tag{2.33}$$

Note that all the generating functions depends on n old variables and n new variables.

제 5 절　보충

- The variation of the action integral between two fixed endpoints:

$$\delta \int_{t_1}^{t_2} L\ dt = \delta \int_{t_1}^{t_2} [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]\ dt = 0$$

- We would like to transform from the old coordinate system $(\mathbf{q},\mathbf{p})$ to a new system $(\mathbf{Q},\mathbf{P})$ with a new Hamiltonian $H'(\mathbf{Q},\mathbf{P},t)$:

$$\delta \int_{t_1}^{t_2} [\mathbf{P}\cdot\dot{\mathbf{Q}} - H'(\mathbf{Q},\mathbf{P},t)]\ dt = 0$$

- One way for both variational integral equalities to be satisfied is to have

$$\lambda[\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)] = \mathbf{P}\cdot\dot{\mathbf{Q}} - H'(\mathbf{Q},\mathbf{P},t) + \frac{dF}{dt}$$

- If $\lambda \neq 1$, it is called the extended canonical transformation. If $\lambda \neq 1$ and $dF/dt = 0$, it is called the scale transformation. These two transformations do not preserve phase space volume

- The function F is in general a function of both the old and new variables as well as the time. We will restrict ourselves to functions that contain half of the old variables and half the new; these are useful for determining the explicit form of the transformation.

  - First type:

$$F = F_1(\mathbf{q},\mathbf{Q},t)$$

$$p_i = +\frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$$

– Second type:

$$F = F_2(\mathbf{q}, \mathbf{P}, t) - \mathbf{Q} \cdot \mathbf{P}$$

$$p_i = +\frac{\partial F_2}{\partial q_i}, \quad Q_i = +\frac{\partial F_2}{\partial P_i}$$

– Third type:

$$F = F_3(\mathbf{p}, \mathbf{Q}, t) + \mathbf{q} \cdot \mathbf{p}$$

$$q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

– Fourth type:

$$F = F_4(\mathbf{p}, \mathbf{P}, t) + \mathbf{q} \cdot \mathbf{p} - \mathbf{Q} \cdot \mathbf{P}$$

$$q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = +\frac{\partial F_4}{\partial P_i}$$

In all cases, new Hamiltonian and equations of motion become:

$$H' = H + \frac{\partial F_i}{\partial t}, \quad \frac{dQ_i}{dt} = \frac{\partial H'}{\partial P_i}, \quad \frac{dP_i}{dt} = -\frac{\partial H'}{\partial Q_i}$$

- Example proof for F_3 case:

$$F = F_3(\mathbf{p}, \mathbf{Q}, t) + \mathbf{q} \cdot \mathbf{p}$$

$$\begin{aligned}
\frac{dF}{dt} &= \frac{\partial F_3}{\partial t} + \sum_i \left(\underbrace{\frac{\partial F_3}{\partial Q_i}}_{=-P_i} \dot{Q}_i + \underbrace{\frac{\partial F_3}{\partial p_i}}_{=-q_i} \dot{p}_i + +q_i \dot{p}_i + p_i \dot{q}_i \right) \\
&= \frac{\partial F_3}{\partial t} + \sum_i \left(-P_i \dot{Q}_i - \cancel{q_i \dot{p}_i} + \cancel{q_i \dot{p}_i} + p_i \dot{q}_i \right) \\
&= \frac{\partial F_3}{\partial t} - \mathbf{P} \cdot \dot{\mathbf{Q}} + \mathbf{p} \cdot \dot{\mathbf{q}}
\end{aligned}$$

Therefore,

$$H' = H + \frac{\partial F_3}{\partial t} \longrightarrow \mathbf{p} \cdot \dot{\mathbf{q}} - H = \mathbf{P} \cdot \dot{\mathbf{Q}} - H' + \frac{dF}{dt}$$

제 6 절 Change the Role of Time Coordinates

Provided that the reference particle moves without backtracking, or some particle coordinate q_j increases in time, we can change the role of that coordinate and time.

$$\int_{t_1}^{t_2} [\mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t)]\ dt = \int_{t_1}^{t_2} [\mathbf{p}\cdot d\mathbf{q} - H(\mathbf{q},\mathbf{p},t)dt]$$

$$\mathbf{p}\cdot d\mathbf{q} - Hdt = \sum_i p_i dq_i + (-H)dt = \left(\sum_{i\neq j} p_i dq_i + (-H)dt\right) - (-p_j)dq_j$$

$$\begin{aligned} H &\longleftrightarrow -p_j \\ t &\longleftrightarrow q_j \end{aligned}$$

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j},\quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \longleftrightarrow \frac{dt}{dq_j} = \frac{\partial(-p_j)}{\partial(-H)},\quad \frac{d(-H)}{dq_j} = -\frac{\partial(-p_j)}{\partial t}$$

제 7 절 숙제

교과서 문제 2.1, 2.2, 2.3, 2.4 를 풀이를 참고로 정리해서 제출 (9월 23일 까지)