Lecture Notes on Accelerator Physics

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Based on book by G. Stupakov and G. Penn

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References

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제 1 장

The basic formulation of mechanics: Lagrangian and Hamiltonian equations of motion

제 1 절 Newton's second law

- Notation:

$$\frac{dx}{dt} = \dot{x}, \quad \frac{dx}{ds} = x'$$

- Basic equations of motion:

$$F = ma$$

or,

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- Simple harmonic oscillator:

$$\ddot{x} + \omega_0^2 x = 0 \tag{1.1}$$

has straightforward solutions of the form

$$a \cos(\omega_0 t + \phi)$$

where a characterizes the amplitude of the motion and ϕ is a phase that describes the timing.

- (Nonlinear) pendulum equation:

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0\,, \tag{1.2}$$

where $\omega_0^2 = g/l$, with l being the length of the pendulum and g being acceleration due to gravity.

- 비선형이지만 위 문제는 다행히 에너지보존으로 해석가능.

$$\frac{1}{2}\frac{d}{dt}\dot{\theta}^2 - \omega_0^2 \frac{d}{dt}\cos\theta = 0\,, \tag{1.3}$$

$$E = \frac{1}{2\omega_0^2}\dot{\theta}^2 - \cos\theta = \text{const} \tag{1.4}$$

is conserved. We call E the energy of the system. Each orbit is characterized by its own energy.

$$\dot{\theta} = \pm\omega_0\sqrt{2(E+\cos\theta)}\,, \tag{1.5}$$

교과서 4.5절 및 그림. 4.4를 볼것.

제 2 절 Lagrangian

How does one write equations of motion for more complicated mechanical systems?

- First step: choosing generalized coordinates

$$q_1, q_2, \ldots, q_n,$$

which uniquely define a snapshot or configuration of the system at a particular time. 여기서 n은 시스템의 자유도.

- Each mechanical system possesses a Lagrangian (function), which depends on the coordinates, velocities $(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$, and time t: $L(q_i, \dot{q}_i, t)$

- Action

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt\,, \tag{1.6}$$

reaches an extremum along the true trajectory of the system varied with fixed end points.

- For mechanical systems, the Lagrangian is equal to the difference between the kinetic and the potential energies. The kinetic energy represents the energy from the particle motion alone, while the potential energy U defines a force acting on the particle through $F = -dU/dx$.

- For a single pendulum with the angle θ as a generalized coordinate q:

$$L(\theta, \dot{\theta}) = \frac{m}{2} l^2 \dot{\theta}^2 + mgl \cos\theta \,. \tag{1.7}$$

- Based on the variational calculus (변분법):

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt = 0 \,, \tag{1.8}$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \,, \qquad i = 1, \ldots, n \,. \tag{1.9}$$

이 상미분 방정식을 풀면 등가적으로 Action을 최소화하는 것이 된다.

- 증명:

$$\begin{aligned}
\delta \int_{t_1}^{t_2} & L(q_i, \dot{q}_i, t) dt \\
&= \int_{t_1}^{t_2} L(q_i + \delta q_i, \dot{q}_i + \dot{\delta} q_i, t) dt - \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt \\
&= \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \\
&= \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt \,,
\end{aligned} \tag{1.10}$$

여기서 $\delta q_i(t_1) = \delta q_i(t_2) = 0$ 이고 마지막 단계에서는 부분적분을 이용.

- Lagrangian is not unique. Lagrangian does not change by adding any function $g(q_i, t)$, which is

$$g(q_i, t) = \frac{df(q_i, t)}{dt} \equiv \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \dot{q}_i \frac{\partial f}{\partial q_i} \,, \tag{1.11}$$

- NOTES
  - We have complete freedom to choose generalized coordinates
  - The Lagrangian formalism is closely related to powerful variational principles
  - There is a connection between the symmetries of the Lagrangian and the conservation laws for the system

$$\frac{\partial L}{\partial q_i} = 0 \rightarrow \frac{\partial L}{\partial \dot{q}_i} = \text{conserved}$$

제 3 절 Lagrangian of a Relativistic Particle in an Electromagnetic Field

For the motion of relativistic charged particles in an electromagnetic field:

$$\begin{aligned} L(\boldsymbol{r}, \boldsymbol{v}, t) &= -mc^2\sqrt{1 - v^2/c^2} + e\boldsymbol{v} \cdot \boldsymbol{A}(\boldsymbol{r}, t) - e\phi(\boldsymbol{r}, t) \\ &= -\frac{mc^2}{\gamma} + e\boldsymbol{v} \cdot \boldsymbol{A}(\boldsymbol{r}, t) - e\phi(\boldsymbol{r}, t)\,, \end{aligned} \tag{1.12}$$

where e is the electric charge of the particle, $\beta = v/c$, and $\gamma = (1-\beta^2)^{-1/2}$ is the Lorentz factor.

- Electric and magnetic fields:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

- In a Cartesian coordinate system:

$$\mathbf{r} = (x, y, z), \quad \mathbf{v} = (v_x, v_y, v_z) = (\dot{x}, \dot{y}, \dot{z})$$

- 연습문제 1.4 와 1.6을 잘 참고할 것.

제 4 절 From Lagrangian to Hamiltonian

The Hamiltonian approach it is simpler to change how a system is characterized in order to suit our needs, and the quantities which come out of this approach have clearer physical meanings as well. These advantages are especially useful in accelerator physics.

- First step: define the generalized momenta p_i

$$p_i(q_k, \dot{q}_k, t) \equiv \frac{\partial L(q_k, \dot{q}_k, t)}{\partial \dot{q}_i}\,, \qquad i = 1, \ldots, n\,. \tag{1.13}$$

- Second step: express all the variables $\dot{q}_i$ in terms of $q_1, q_2, \ldots q_n, p_1, p_2, \ldots, p_n$ and t

$$\dot{q}_i = \dot{q}_i(q_k, p_k, t)\,, \qquad i = 1, \ldots, n\,. \tag{1.14}$$

- Third step: construct the Hamiltonian function

$$H = \sum_{i=1}^{n} p_i\dot{q}_i - L(q_k, \dot{q}_k, t)\,, \tag{1.15}$$

- The equation of motion of the system:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}\,. \tag{1.16}$$

Here, the variables p_i and q_i are called canonically conjugate pairs of variables.

- 증명:

$$-\frac{\partial H}{\partial q_i} = -\frac{\partial}{\partial q_i}\left(\sum_{k=1}^{n} p_k \dot{q}_k - L\right) = \sum_{k=1}^{n}\left(-p_k \frac{\partial \dot{q}_k}{\partial q_i} + \frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q_i}\right) + \frac{\partial L}{\partial q_i}\,. \tag{1.17}$$

$$-\frac{\partial H}{\partial q_i} = \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{dp_i}{dt}\,. \tag{1.18}$$

- Hamiltonian for the pendulum:

$$H(\theta, p) = \frac{p^2}{2ml^2} - \omega_0^2 m l^2 \cos\theta\,, \tag{1.19}$$

The generalized momentum p corresponding to the angular variable θ is $p = ml^2\dot{\theta}$.

제 5 절 Hamiltonian of a Charged Particle in an Electromagnetic Field

- Notation: the canonical conjugate momentum $\boldsymbol{\pi} = (\pi_x, \pi_y, \pi_z)$

$$\boldsymbol{\pi} = \frac{\partial L}{\partial \boldsymbol{v}} = m\frac{\boldsymbol{v}}{\sqrt{1 - v^2/c^2}} + e\boldsymbol{A} = m\gamma\boldsymbol{v} + e\boldsymbol{A}\,. \tag{1.20}$$

Note that the conjugate momentum $\boldsymbol{\pi}$ differs from the kinetic momenta $\gamma m\mathbf{v} = \gamma m\boldsymbol{\beta}c$ of the particle.

- The Hamiltonian is derived as

$$\begin{aligned} H &= \boldsymbol{v}\cdot\boldsymbol{\pi} - L \\ &= \boldsymbol{v}\cdot\boldsymbol{\pi} + mc^2\sqrt{1 - v^2/c^2} - e\boldsymbol{v}\cdot\boldsymbol{A} + e\phi \\ &= m\gamma v^2 + \frac{mc^2}{\gamma} + e\phi \\ &= m\gamma c^2 + e\phi\,. \end{aligned} \tag{1.21}$$

The Hamiltonian is the sum of the particle energy and the potential energy associated with the electrostatic potential.

- Using

$$\gamma^2 = 1 + \gamma^2\beta^2, \tag{1.22}$$

$$\gamma^2\beta^2 = \frac{(\boldsymbol{\pi} - e\boldsymbol{A})^2}{m^2c^2}, \tag{1.23}$$

we obtain

$$\gamma^2 = 1 + \frac{(\boldsymbol{\pi} - e\boldsymbol{A})^2}{m^2c^2}, \tag{1.24}$$

and

$$H(\boldsymbol{r}, \boldsymbol{\pi}, t) = \sqrt{(mc^2)^2 + c^2(\boldsymbol{\pi} - e\boldsymbol{A}(\boldsymbol{r}, t))^2} + e\phi(\boldsymbol{r}, t)\,. \tag{1.25}$$

Therefore, the vector potential is also present in the Hamiltonian.

제 6 절 The Poisson Bracket

- $f(q_i, p_i, t) = f(q_i(t), p_i(t), t)$ can be considered as a function of time t only.
- Derivative with respect to time (convective or Lagrangian derivative): chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right). \tag{1.26}$$

- Using Hamilton's equations of motion

$$\begin{aligned} \frac{df}{dt}_{\text{conv}} &= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \\ &= \frac{\partial f}{\partial t} + \{f, H\}\,, \end{aligned} \tag{1.27}$$

- Poisson bracket:

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right), \tag{1.28}$$

- Poisson bracket is anti-commutative

$$\{g, f\} = -\{f, g\}\,. \tag{1.29}$$

$$\{f, f\} = 0\,. \tag{1.30}$$

- If $df/dt = 0$, then f is an integral of motion,meaning that its value remains constant along the orbit. Note that if f does not explicitly depend on time, $f = f(q_i, p_i)$, it is an integral of motion if and only if $\{f, H\} = 0$.

- A Hamiltonian that does not depend explicitly on time is an integral of motion, because the Poisson bracket of H with itself is always equal to zero ($dH/dt = \{H, H\} = 0$).

- Each coordinate q_k and momentum p_k are considered as a function that only depends on itself
$$\frac{\partial q_k}{\partial q_i} = \frac{\partial p_k}{\partial p_i} = \delta_{ik}, \quad \frac{\partial q_k}{\partial p_i} = \frac{\partial p_k}{\partial q_i} = 0. \tag{1.31}$$

$$\{q_i, q_k\} = \{p_i, p_k\} = 0\,, \qquad \{q_i, p_k\} = \delta_{ik}\,. \tag{1.32}$$

제 7 절 숙제

- 교과서 문제 1.4 와 1.6 을 풀이를 참고로 정리해서 제출 (9월 17일까지)