Fitting a Step Function to a Point Set

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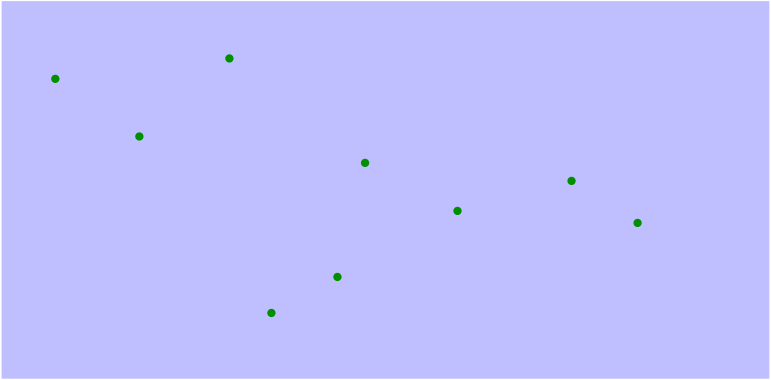
Problem statement



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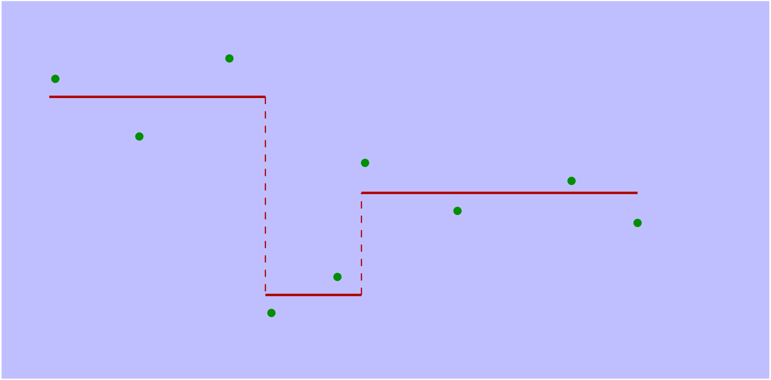

- INPUT: a set P of n points and an integer k.
- OUTPUT: a step function f^* with k steps that minimizes the maximum vertical distance $\varepsilon^* = d(P, f^*)$ between f and P.
- $d(f, P) = \max_i |f(x_i) - y_i|$ when $P = \{(x_1, y_1) \ldots (x_n, y_n)\}$

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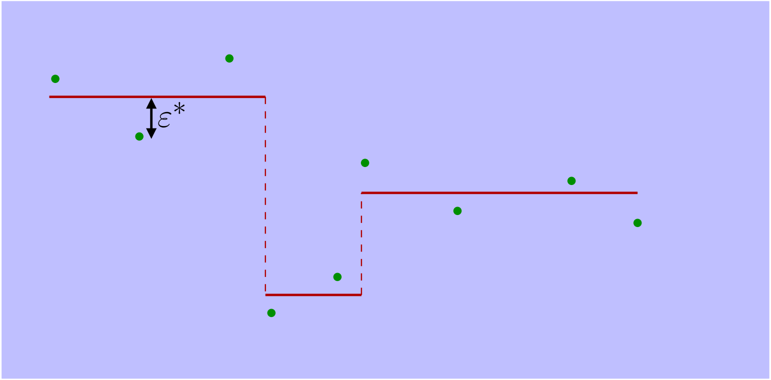

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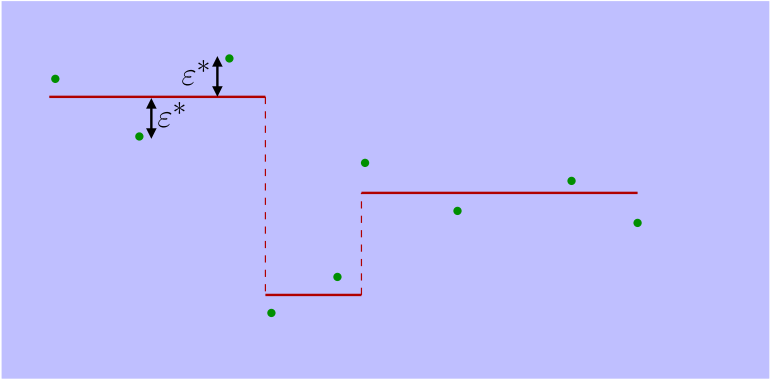

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  - based on Frederickson's path partitioning algorithm, and the data structure by Gabow, Bentley, and Tarjan for range maxima.
- An $O(n \log^4 n)$ time algorithm for the weighted case.
- An $O(n \log n \cdot h^2)$ time algorithm when h outliers are allowed.

Previous work

- Diáz-Báñez and Mesa (2001): $O(n^2 \log n)$.

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- In databases, the problem is known as the problem of computing a *Maximum Error Histogram*.

Decision algorithm

- Decision problem: given P, k, $\varepsilon > 0$, is there a k-step function f such that $d(P, f) \leqslant \varepsilon$?

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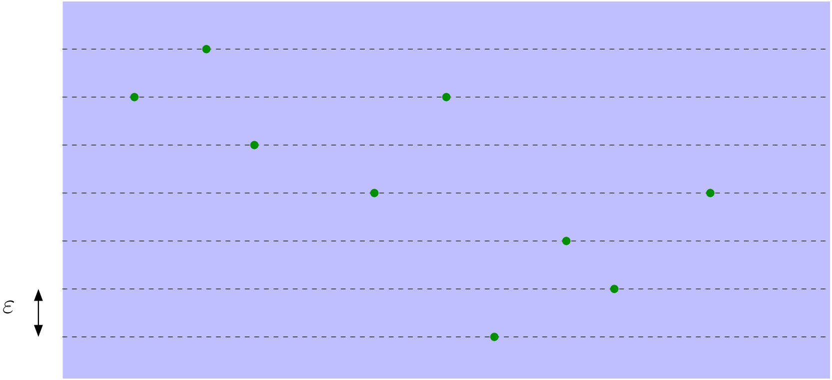

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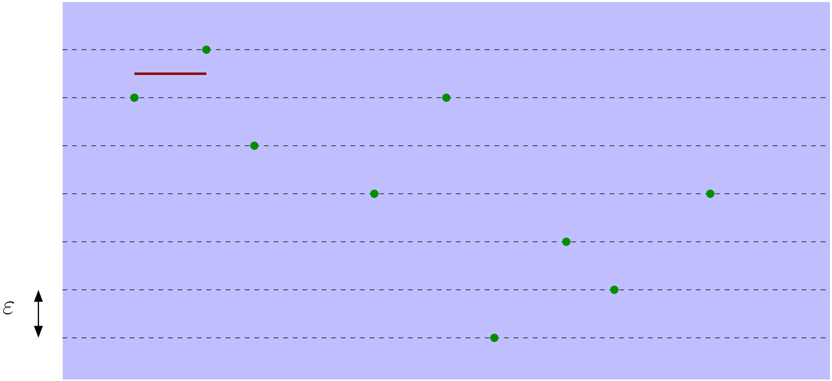

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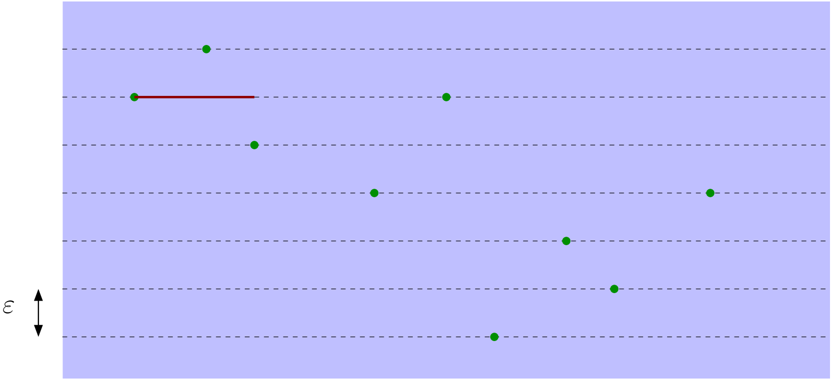

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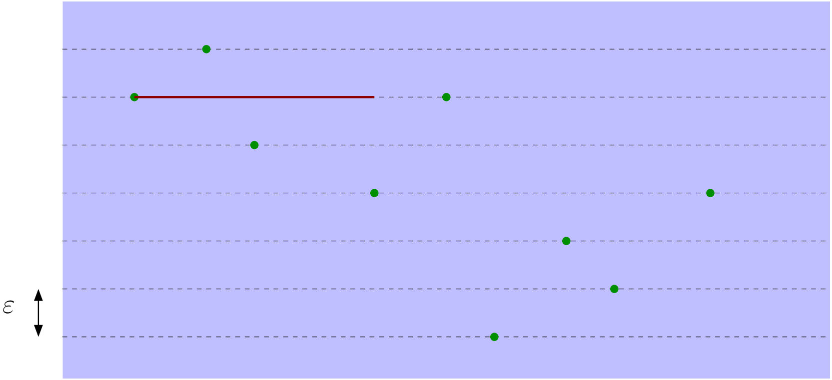

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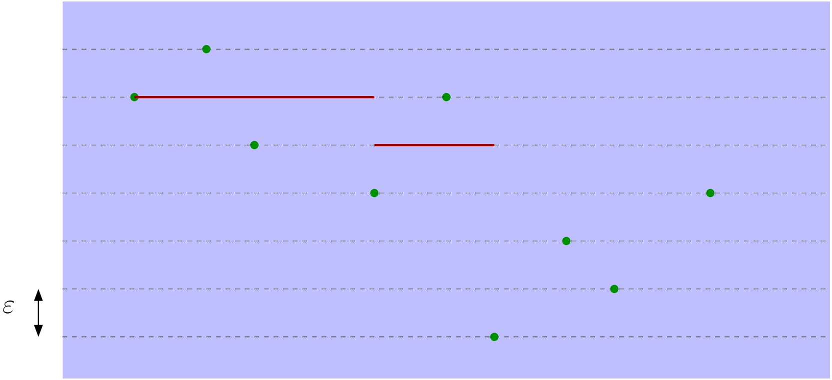

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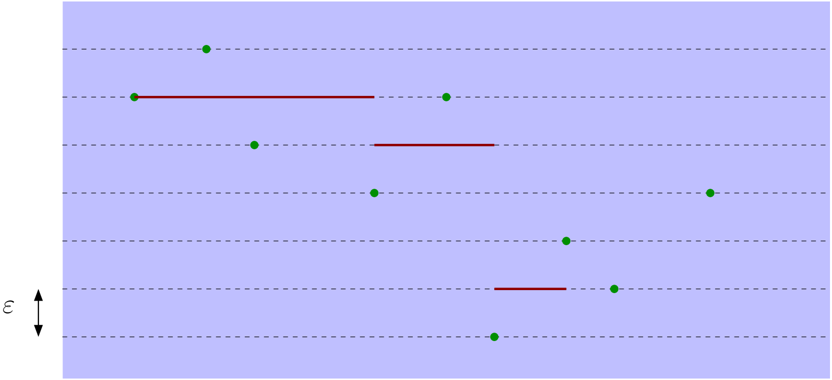

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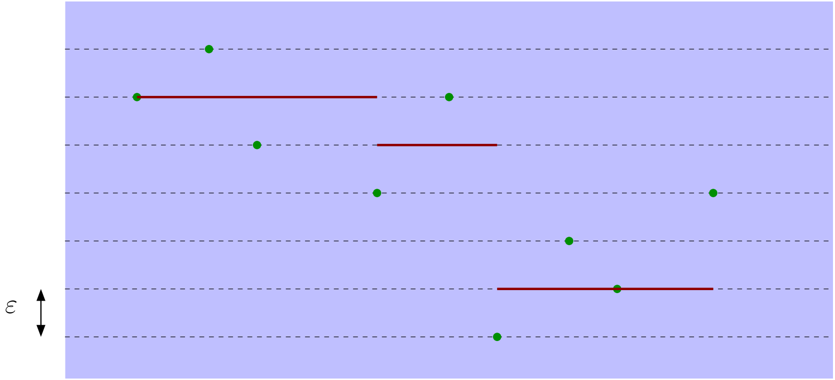

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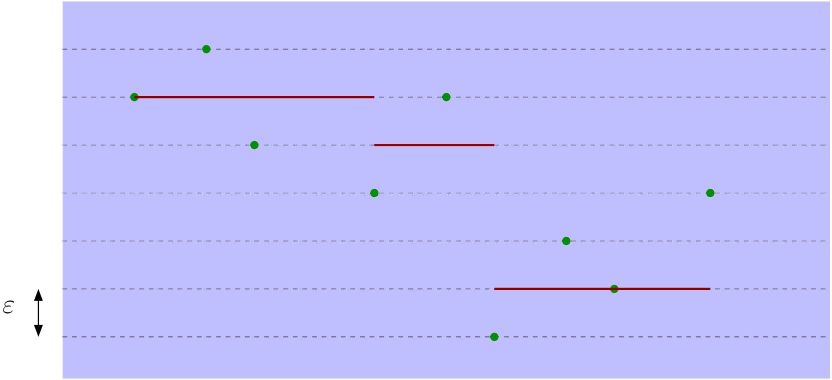

- FALSE if $k = 3$

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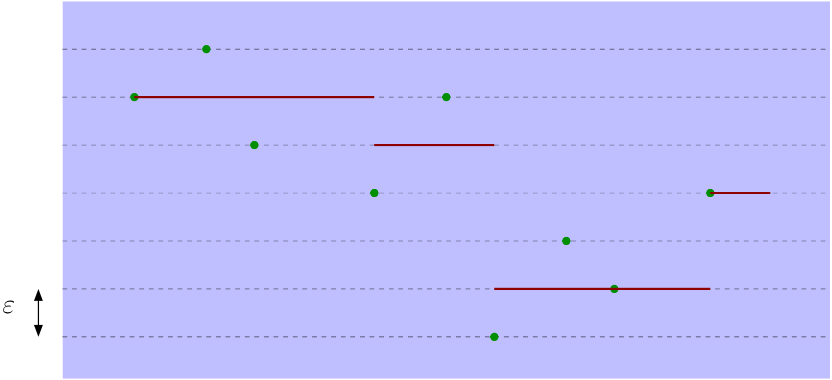

- FALSE if $k = 3$
- TRUE if $k = 4$

Optimization algorithm

- $\{y_1, \ldots, y_n\}$ denote the y-coordinates of points in P.

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- Let $\tilde{y}_1 \leqslant \ldots \leqslant \tilde{y}_n$ denote the y-coordinates in sorted order.
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- Let $\tilde{y}_1 \leqslant \ldots \leqslant \tilde{y}_n$ denote the y-coordinates in sorted order.
- Let $M_{ij} = \frac{1}{2}\left(\tilde{y}_i - \tilde{y}_{n+1-j}\right)$.
- M is a *sorted matrix*: $i \leqslant i'$ and $j \leqslant j'$ implies $M_{ij} \leqslant M_{i'j'}$.

Searching in a sorted matrix

$$\begin{matrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
3 & 6 & 7 & 8 & 10 & 11 & 12 & 13 \\
4 & 7 & 8 & 9 & 11 & 12 & 13 & 15 \\
6 & 9 & 10 & 11 & 12 & 14 & 15 & 17 \\
7 & 10 & 11 & 13 & 14 & 16 & 17 & 18 \\
8 & 12 & 13 & 14 & 15 & 18 & 19 & 20 \\
9 & 13 & 14 & 16 & 17 & 19 & 21 & 23
\end{matrix}$$

- a sorted matrix M

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\end{matrix}$$

- a sorted matrix M
- an increasing boolean function g: there exists x^* such that $g(x) = \mathit{FALSE}$ for all $x < x^*$, and $g(x) = \mathit{TRUE}$ for all $x \geqslant x^*$.

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1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
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\end{matrix}$$

- a sorted matrix M
- an increasing boolean function g: there exists x^* such that $g(x) = \mathit{FALSE}$ for all $x < x^*$, and $g(x) = \mathit{TRUE}$ for all $x \geqslant x^*$.
- Problem: search for x^* in M.

Searching in a sorted matrix

$$
\begin{array}{cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
3 & 6 & 7 & 8 & 10 & 11 & 12 & 13 \\
4 & 7 & 8 & 9 & 11 & 12 & 13 & 15 \\
\hline
6 & 9 & 10 & 11 & 12 & 14 & 15 & 17 \\
7 & 10 & 11 & 13 & 14 & 16 & 17 & 18 \\
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\end{array}
$$

- smallest elements in submatrices: $\{1, 5, 6, 12\}$; median=6.

Searching in a sorted matrix

$$
\begin{array}{cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
3 & 6 & 7 & 8 & 10 & 11 & 12 & 13 \\
4 & 7 & 8 & 9 & 11 & 12 & 13 & 15 \\
\hline
6 & 9 & 10 & 11 & 12 & 14 & 15 & 17 \\
7 & 10 & 11 & 13 & 14 & 16 & 17 & 18 \\
8 & 12 & 13 & 14 & 15 & 18 & 19 & 20 \\
9 & 13 & 14 & 16 & 17 & 19 & 21 & 23
\end{array}
$$

- smallest elements in submatrices: $\{1, 5, 6, 12\}$; median=6.
- largest elements in submatrices: $\{9, 15, 16, 23\}$; median=16.

Searching in a sorted matrix

$$\begin{array}{cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
3 & 6 & 7 & 8 & 10 & 11 & 12 & 13 \\
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\end{array}$$

- smallest elements in submatrices: $\{1, 5, 6, 12\}$; median=6.
- largest elements in submatrices: $\{9, 15, 16, 23\}$; median=16.
- we compute $g(6) = \mathit{FALSE}$ and $g(16) = \mathit{TRUE}$.

Searching in a sorted matrix

$$\begin{array}{cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
3 & 6 & 7 & 8 & 10 & 11 & 12 & 13 \\
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\end{array}$$

- smallest elements in submatrices: $\{1, 5, 6, 12\}$; median=6.
- largest elements in submatrices: $\{9, 15, 16, 23\}$; median=16.
- we compute $g(6) = \mathit{FALSE}$ and $g(16) = \mathit{TRUE}$.
- we did not make progress.

Searching in a sorted matrix

$$\begin{array}{cc|cc|cc|cc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
3 & 6 & 7 & 8 & 10 & 11 & 12 & 13 \\
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\end{array}$$

- smallest elements: $\{1, 3, 5, 6, 7, 8, 10, 12, 13, 15, 19\}$; median$=8$.

Searching in a sorted matrix

$$
\begin{array}{cc|cc|cc|cc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
3 & 6 & 7 & 8 & 10 & 11 & 12 & 13 \\
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- smallest elements: $\{1, 3, 5, 6, 7, 8, 10, 12, 13, 15, 19\}$; median=8.
- largest elements: $\{3, 6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 19, 23\}$; median=12.

Searching in a sorted matrix

$$\begin{array}{cc|cc|cc|cc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
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\end{array}$$

- smallest elements: $\{1, 3, 5, 6, 7, 8, 10, 12, 13, 15, 19\}$; median=8.
- largest elements: $\{3, 6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 19, 23\}$; median=12.
- we compute $g(8) = \mathit{FALSE}$ and $g(12) = \mathit{TRUE}$.

Searching in a sorted matrix

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\begin{array}{cc|cc|cc|cc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 \\
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\end{array}
$$

- smallest elements: $\{1, 3, 5, 6, 7, 8, 10, 12, 13, 15, 19\}$; median=8.
- largest elements: $\{3, 6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 19, 23\}$; median=12.
- we compute $g(8) = \mathit{FALSE}$ and $g(12) = \mathit{TRUE}$.
- we can discard 8 submatrices.

Searching in a sorted matrix

						7	8
						9	10
		7	8	10	11	12	13
		8	9	11	12	13	15
6	9	10	11	12	14		
7	10	11	13	14	16		
8	12						
9	13						

- smallest elements: $\{1, 3, 5, 6, 7, 8, 10, 12, 13, 15, 19\}$; median=8.
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6	9	10	11	12	14		
7	10	11	13	14	16		
8	12						
9	13						

- remaining elements $\{7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$; median=12.

Searching in a sorted matrix

						7	8
						9	10
		7	8	10	11	12	13
		8	9	11	12	13	15
6	9	10	11	12	14		
7	10	11	13	14	16		
8	12						
9	13						

- remaining elements $\{7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$; median=12.
- we compute $g(12) = \mathit{TRUE}$

Searching in a sorted matrix

						7	8
						9	10
		7	8	10	11	12	13
		8	9	11	12	13	15
6	9	10	11	12	14		
7	10	11	13	14	16		
8	12						
9	13						

- remaining elements $\{7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$; median=12.
- we compute $g(12) = TRUE$
- we discard all elements > 12.

Searching in a sorted matrix

						7	8
						9	10
		7	8	10	11	12	
		8	9	11	12		
6	9	10	11	12			
7	10	11					
8	12						
9							

- repeat until one value is left.

Searching in a sorted matrix

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- This algorithm works in the following, more general case:

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  - $\pi(n) = O(n)$ and $\kappa(n) = O(1)$, so running time $O(n)$.

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- Conclusion: the sorted case can be solved in $O(n)$ time.

Weighted version

- Given a set of points in the plane $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ with positive weights $\mu_1, \ldots, \mu_n$ and an integer $k > 0$, compute a k-step function f such that

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- L_1-problem: minimize the sum of the vertical distances between P and f.