

Lower Bounds for Geometric Diameter Problems

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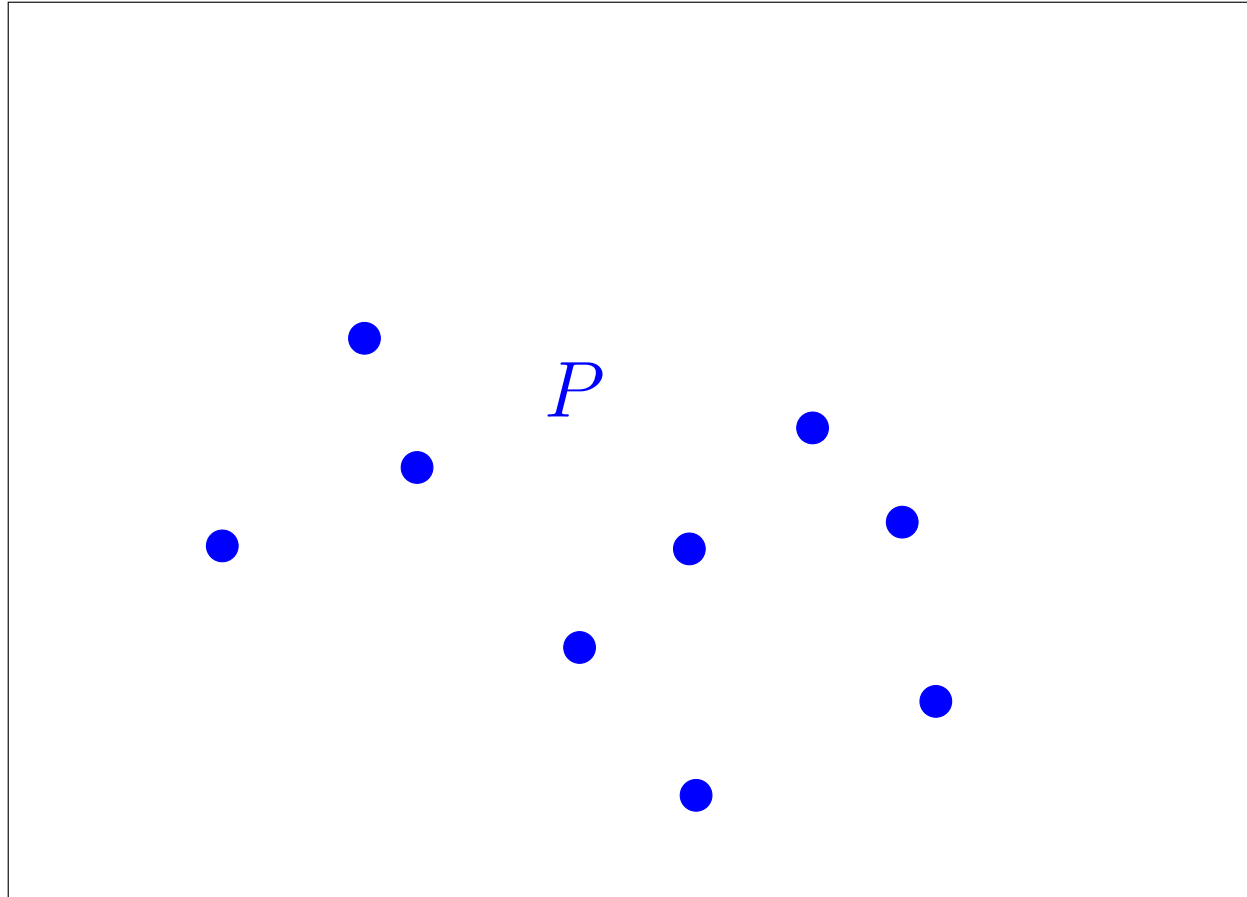
INRA Jouy-en-Josas

Outline

- Review of previous work on the 2D and 3D diameter problems.
- $\Omega(n \log n)$ lower bound for computing the diameter of a 3D convex polytope.
- Reduction from Hopcroft's problem to the diameter problem for point sets in \mathbb{R}^7 .

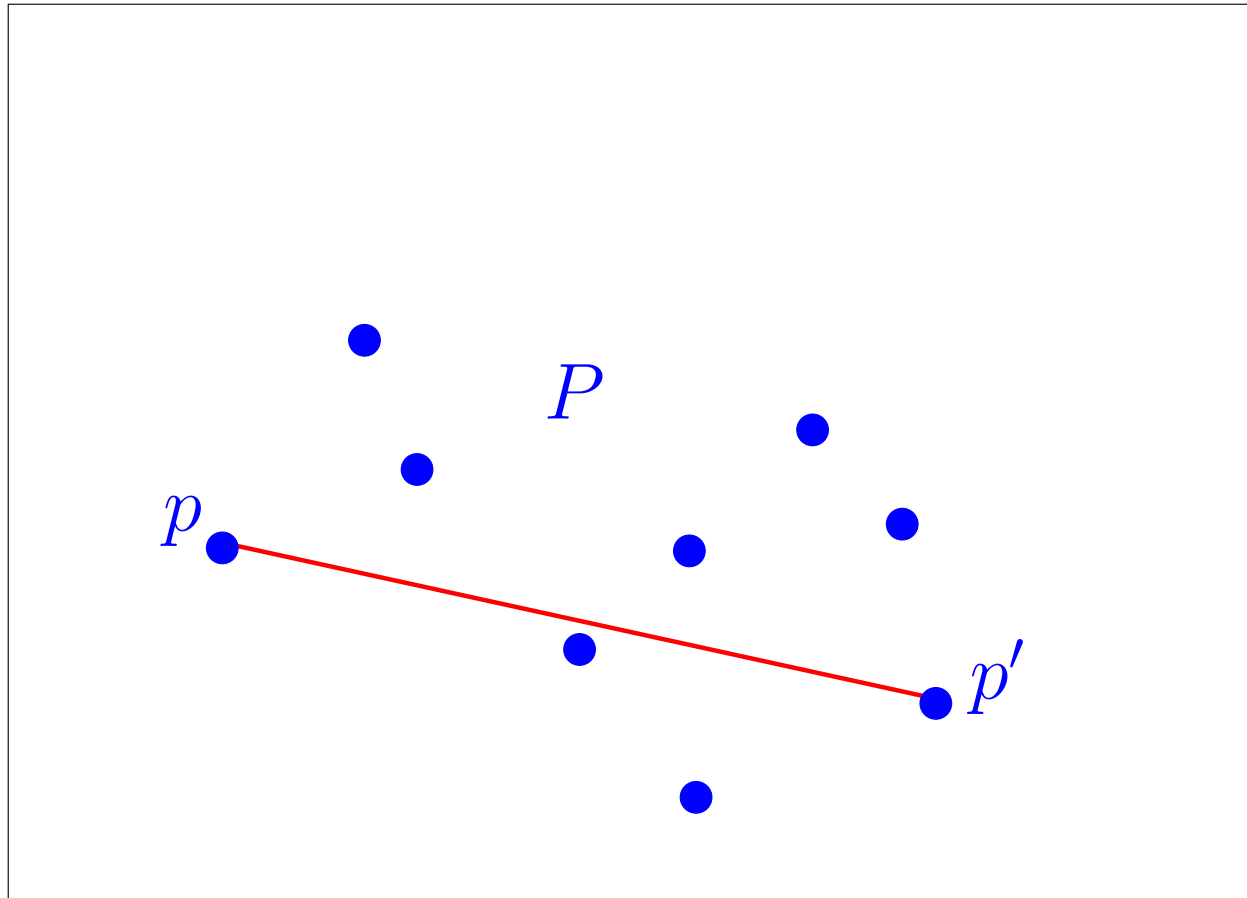
Previous work

The diameter problem



- INPUT: a set P of n points in \mathbb{R}^d .
- OUTPUT: $\text{diam}(P) := \max\{d(x, y) \mid x, y \in P\}$.

The diameter problem

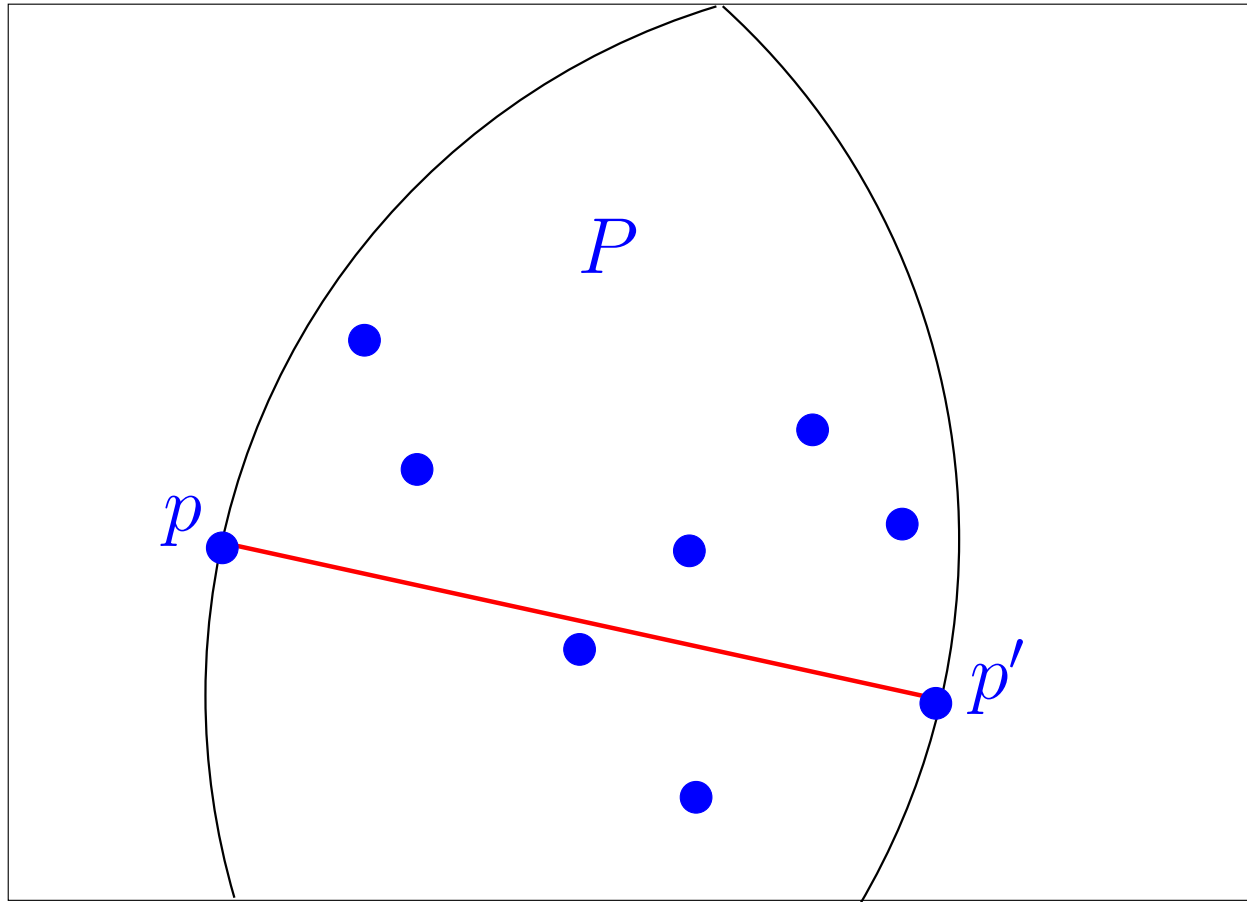


- $\text{diam}(P) = d(p, p')$.

Decision problem

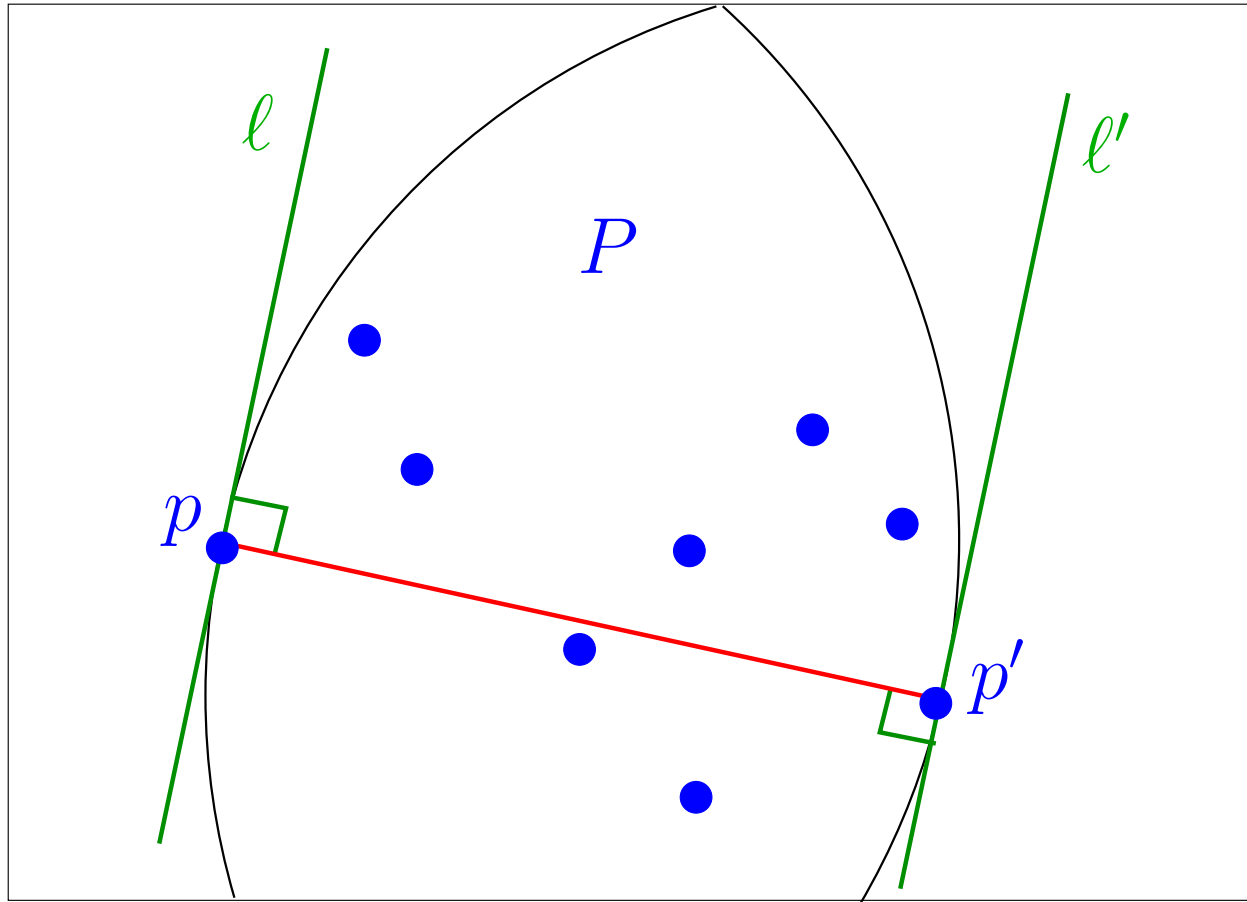
- We will give lower bounds for the *decision problem* associated with the diameter problem.
- INPUT: a set P of n points in \mathbb{R}^d .
- OUTPUT:
 - YES if $\text{diam}(P) < 1$
 - NO if $\text{diam}(P) \geq 1$

Observation



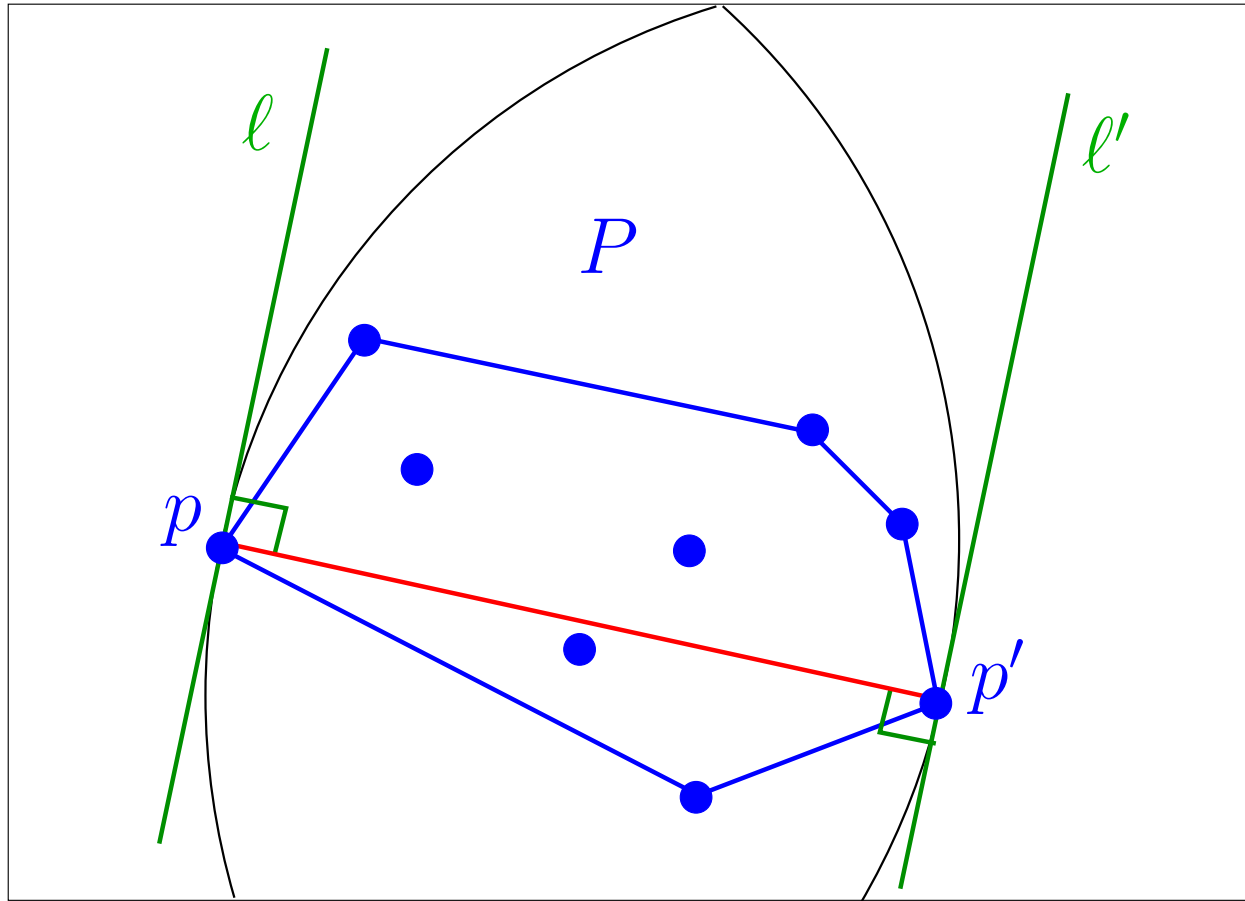
- P lies in the intersection of the two balls with radius $d(p, p')$ centered at p and p' .

The diameter problem



- P lies between two parallel hyperplanes through p and p' . We say that (p, p') is an *antipodal pair*.

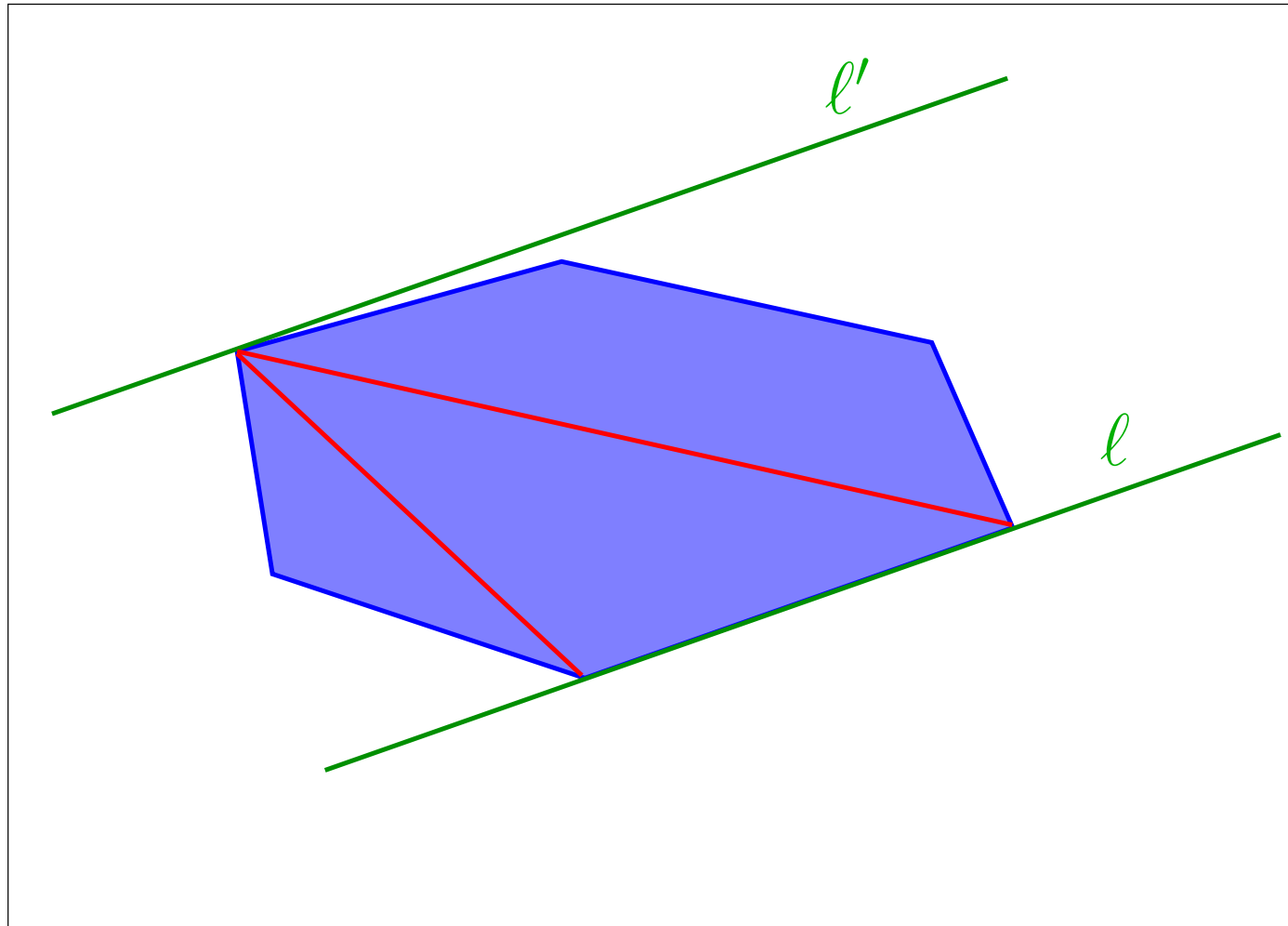
The diameter problem



- Any antipodal pair (and therefore any diametral pair) lies on the convex hull $CH(P)$ of P .

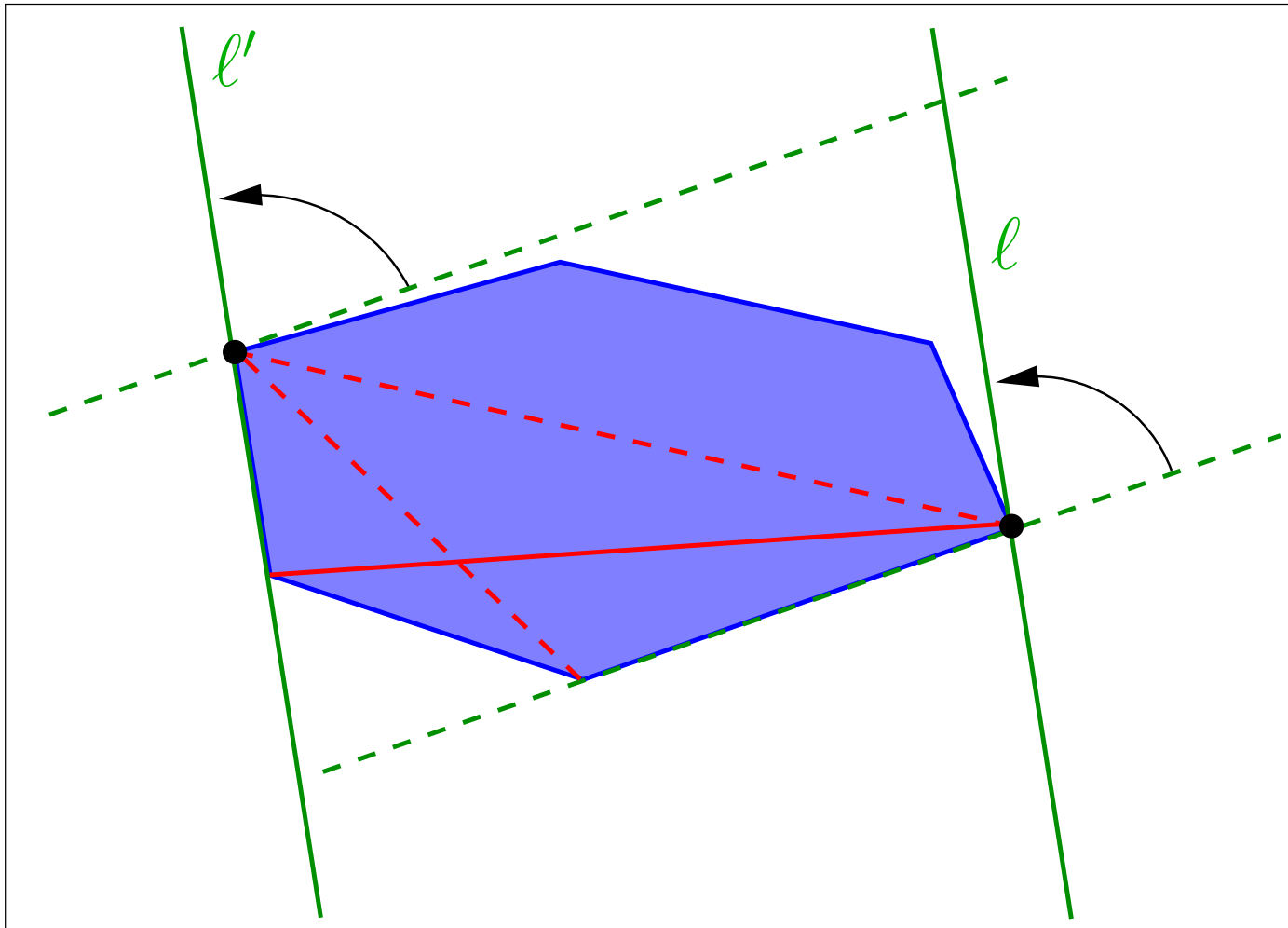
Finding the antipodal pairs

- The rotating calipers technique.



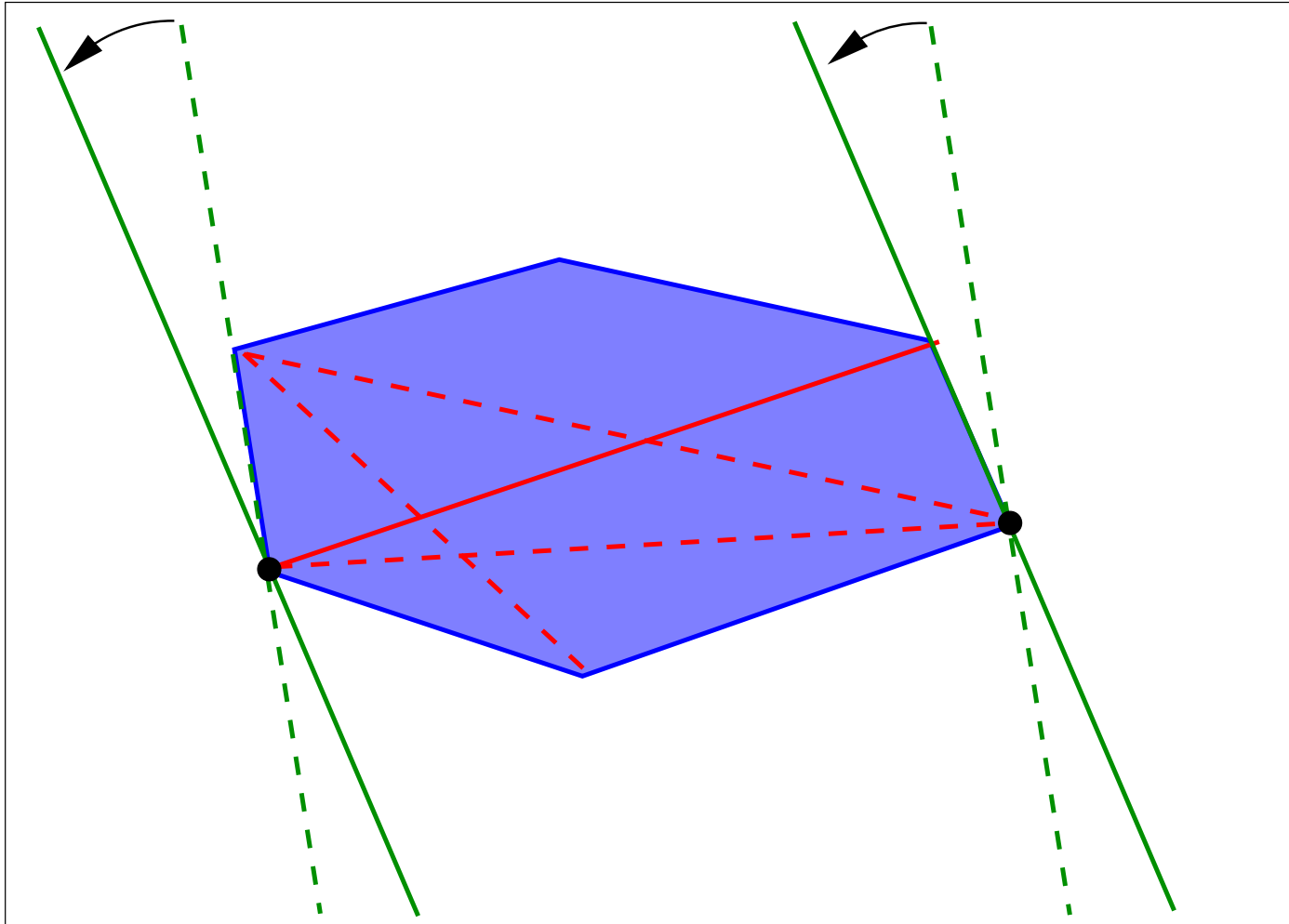
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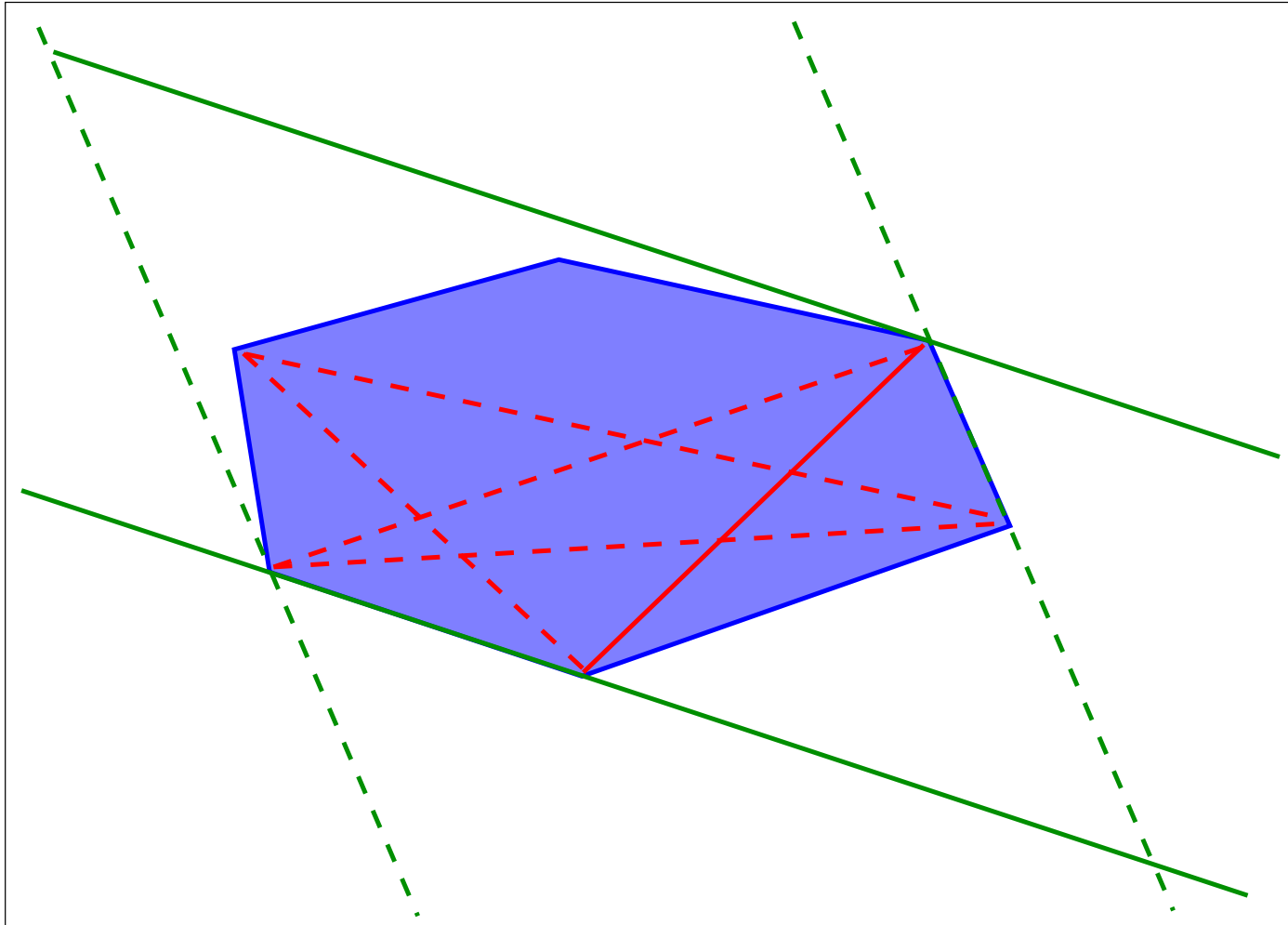
Finding the antipodal pairs

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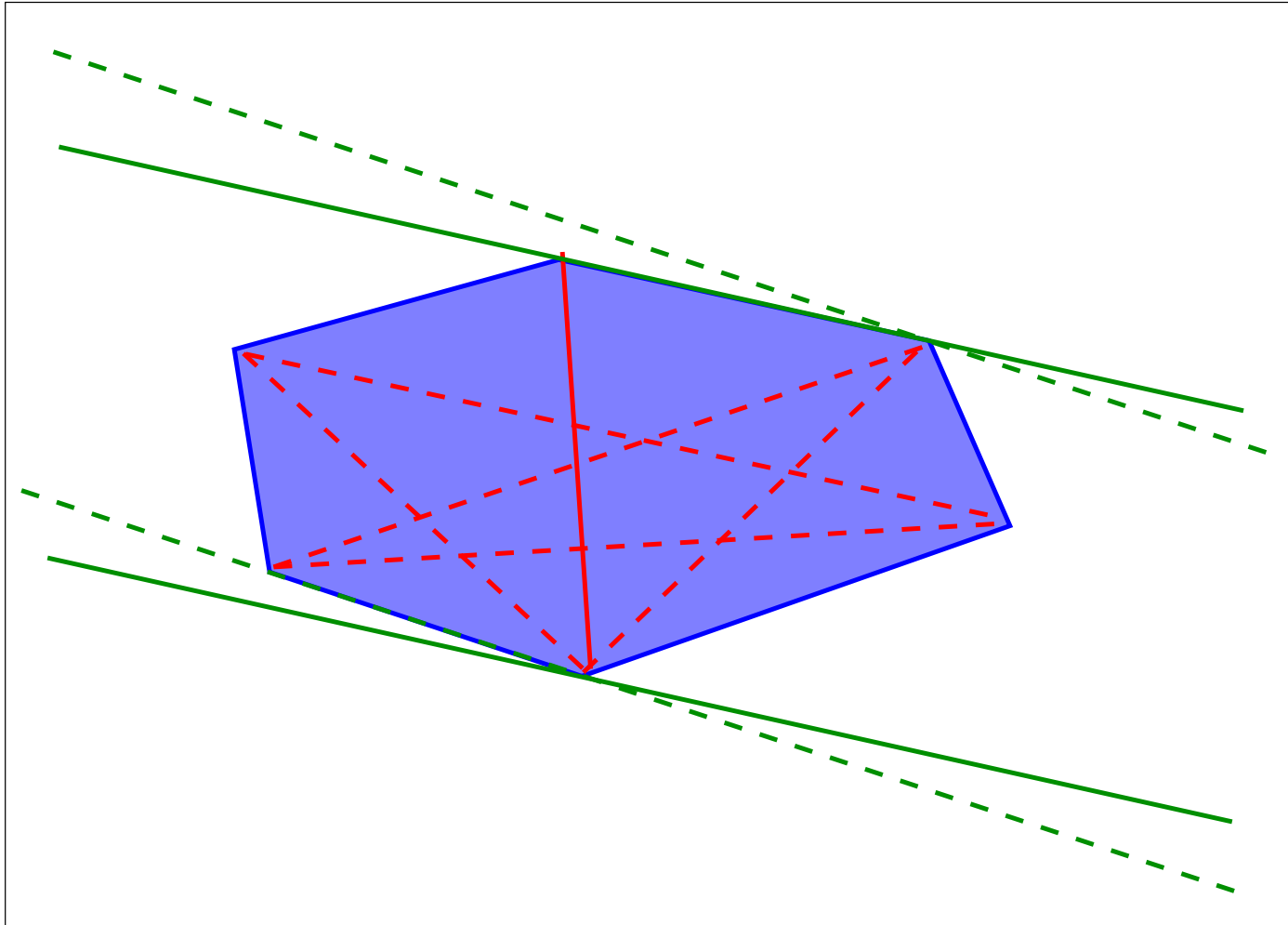
Finding the antipodal pairs

- The rotating calipers technique.



Finding the antipodal pairs

- The rotating calipers technique.



Computing the diameter of a 2D-point set

- Compute the convex hull $CH(P)$ of P .
 - $O(n \log n)$ time.
- Find all the antipodal pairs on $CH(P)$.
 - There are at most n such pairs in non-degenerate cases.
 - $O(n)$ time using the rotating calipers technique.
- Find the diametral pairs among the antipodal pairs.
 - $O(n)$ time by brute force.
- Conclusion:
 - The diameter of a 2D-point set can be found in $O(n \log n)$ time
 - The diameter of a convex polygon can be found in $O(n)$ time.

3D-diameter problem

- Randomized $O(n \log n)$ time algorithm (Clarkson and Shor, 1988).
 - Randomized incremental construction of an intersection of balls and decimation.
- Deterministic $O(n \log n)$ time algorithm (E. Ramos, 2000).
- These two algorithms compute the diameter of an n -point set in \mathbb{R}^3 .
- Can we compute the diameter of a convex 3D-polytope in linear time?
 - No, we give an $\Omega(n \log n)$ lower bound.

Model of computation

Real-RAM

- Real Random Access Machine.
- Each registers stores a *real* number.
- Access to registers in unit time.
- Arithmetic operation ($+$, $-$, \times , $/$) in unit time.

Algebraic computation tree: definition

- Input: $\bar{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.
- Output: YES or NO
- It is a binary tree with 3 types of nodes
 - **Leaves**: YES or NO
 - Degree-1 nodes: **computation nodes**. Perform $\{+, -, \times, /, \sqrt{\cdot}\}$ on two operands. An operand is
 - a real constant,
 - some input number x_i , or
 - the value obtained by computation nodes that are ancestors of the current node.
 - Degree-2 nodes: **branching nodes**. Compares with 0 the value obtained at a computation node that is an ancestor of the current node.

Algebraic computation tree (ACT)

- We say that an ACT *decides* $S \subset \mathbb{R}^n$ if
 - $\forall (x_1, \dots, x_n) \in S$, it reaches a leaf labeled YES, and
 - $\forall (x_1, \dots, x_n) \notin S$, it reaches a leaf labeled NO.
- The ACT model is stronger than the real-RAM model.
- To get a lower bound on the worst-case running time of a real-RAM that decides S , it suffices to have a lower bound on the *depth* of all the ACTs that decide S

Theorem 1 (Ben-Or) *Any ACT that decides S has depth*

$$\Omega(\log(\text{number of connected components of } S)).$$

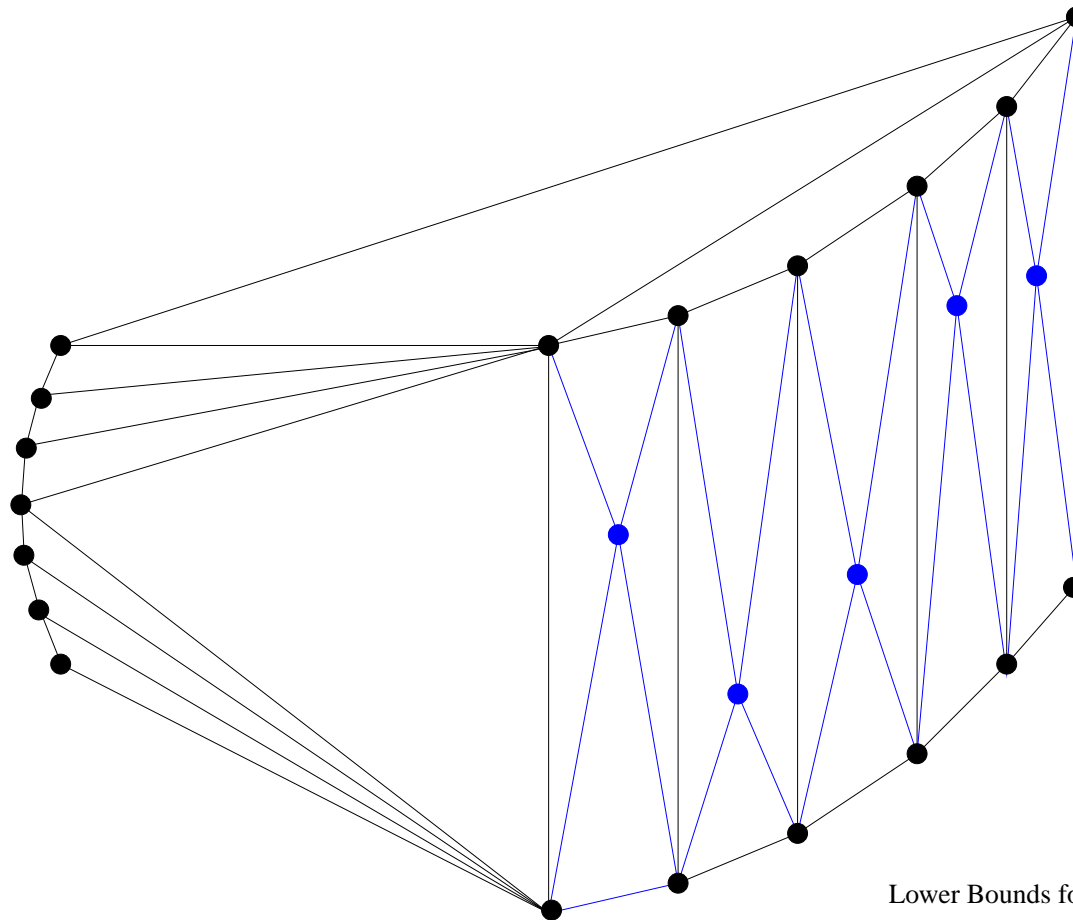
Lower bound for 3D convex polytopes

Problem statement

- We are given a convex 3-polytope P with n vertices.
- P is given by the coordinates of its vertices and its *combinatorial structure*:
 - All the inclusion relations between its vertices, edges and faces.
 - The cyclic ordering of the edges of each face.
- Example: P is given in a doubly-connected edge list.
- Problem: we want to decide whether $\text{diam}(P) < 1$.
- We show an $\Omega(n \log n)$ lower bound. Our approach:
 - We define a family of convex polytopes.
 - We show that the sub-family with diameter < 1 has $n^{\Omega(n)}$ connected components.
 - We apply Ben-Or's bound.

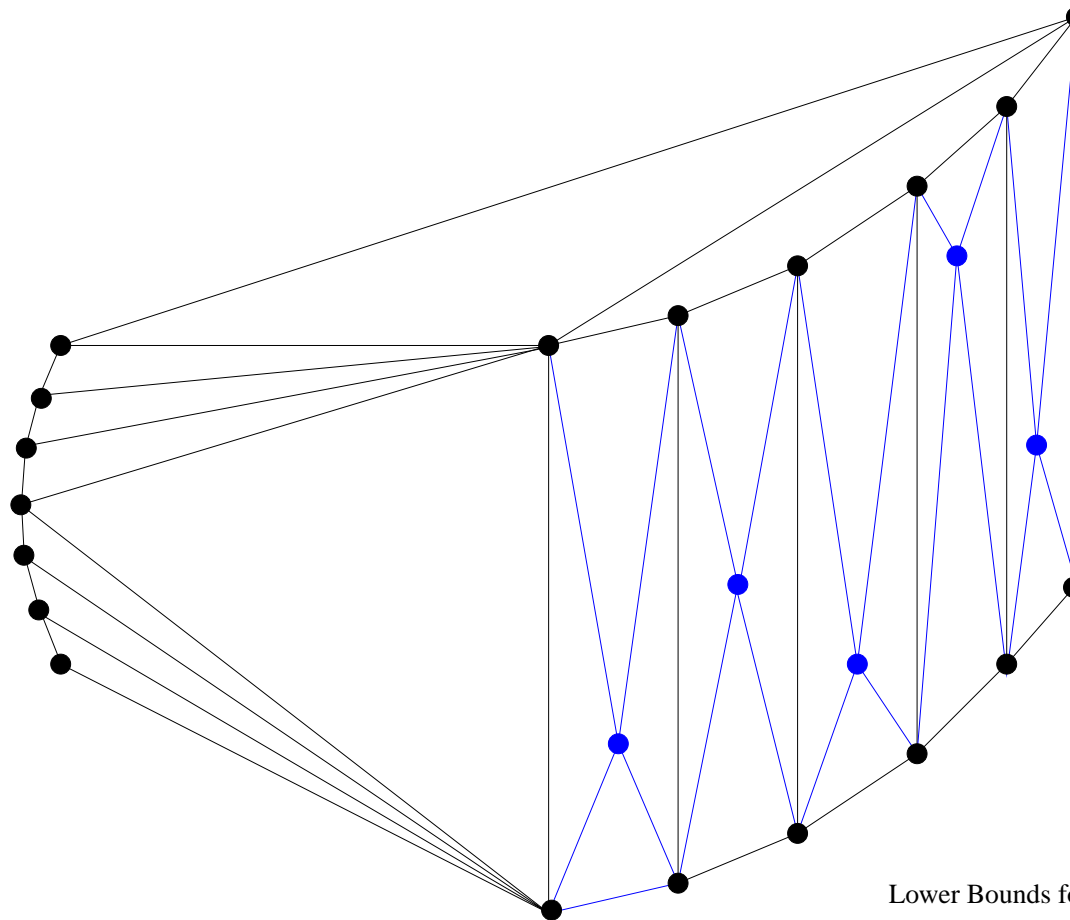
Polytopes $P(\bar{\beta})$

- The family of polytopes is parametrized by $\bar{\beta} \in \mathbb{R}^{2n-1}$.
- When n is fixed, only the $2n - 1$ blue points change with $\bar{\beta}$.



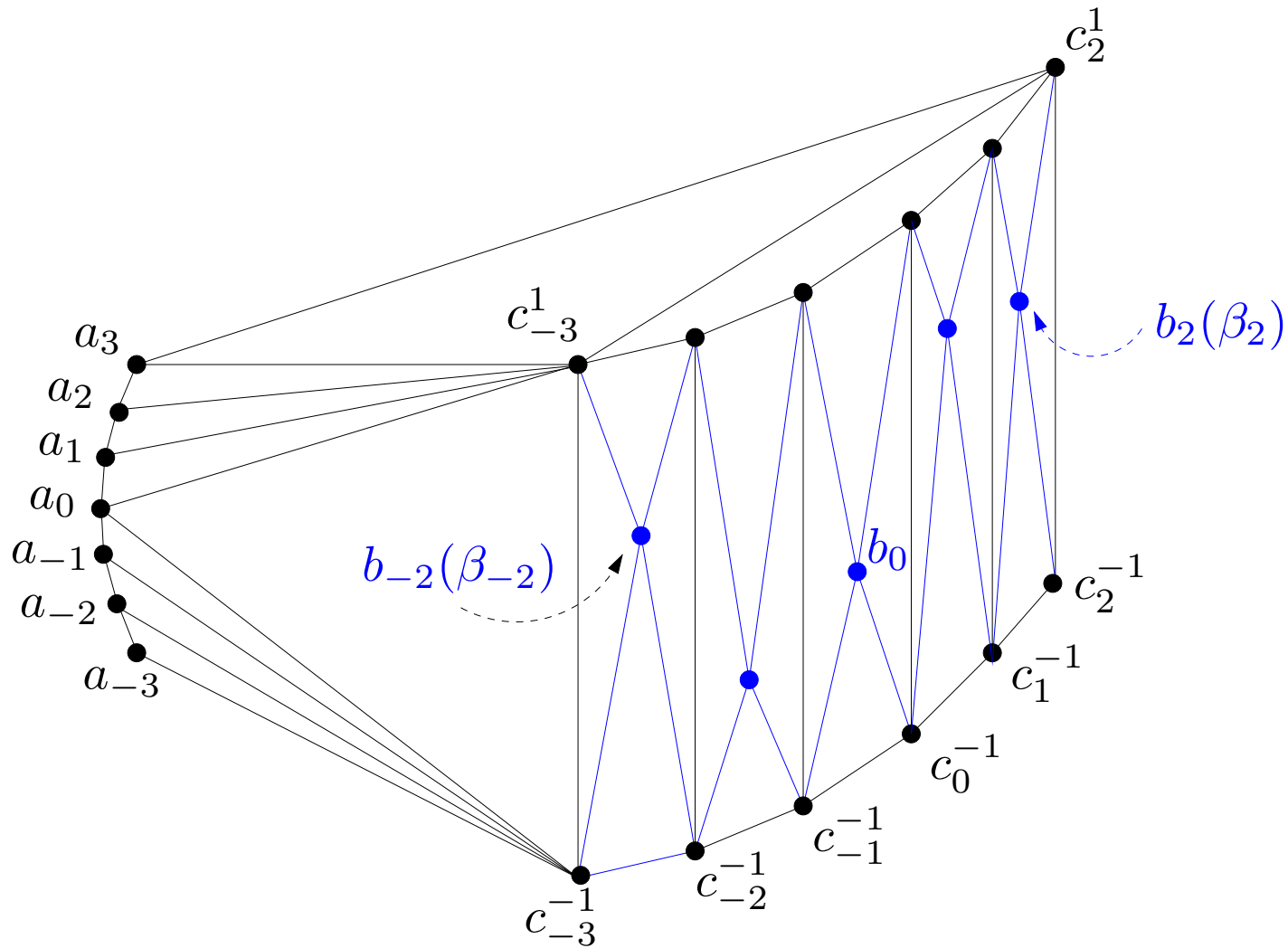
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Notation

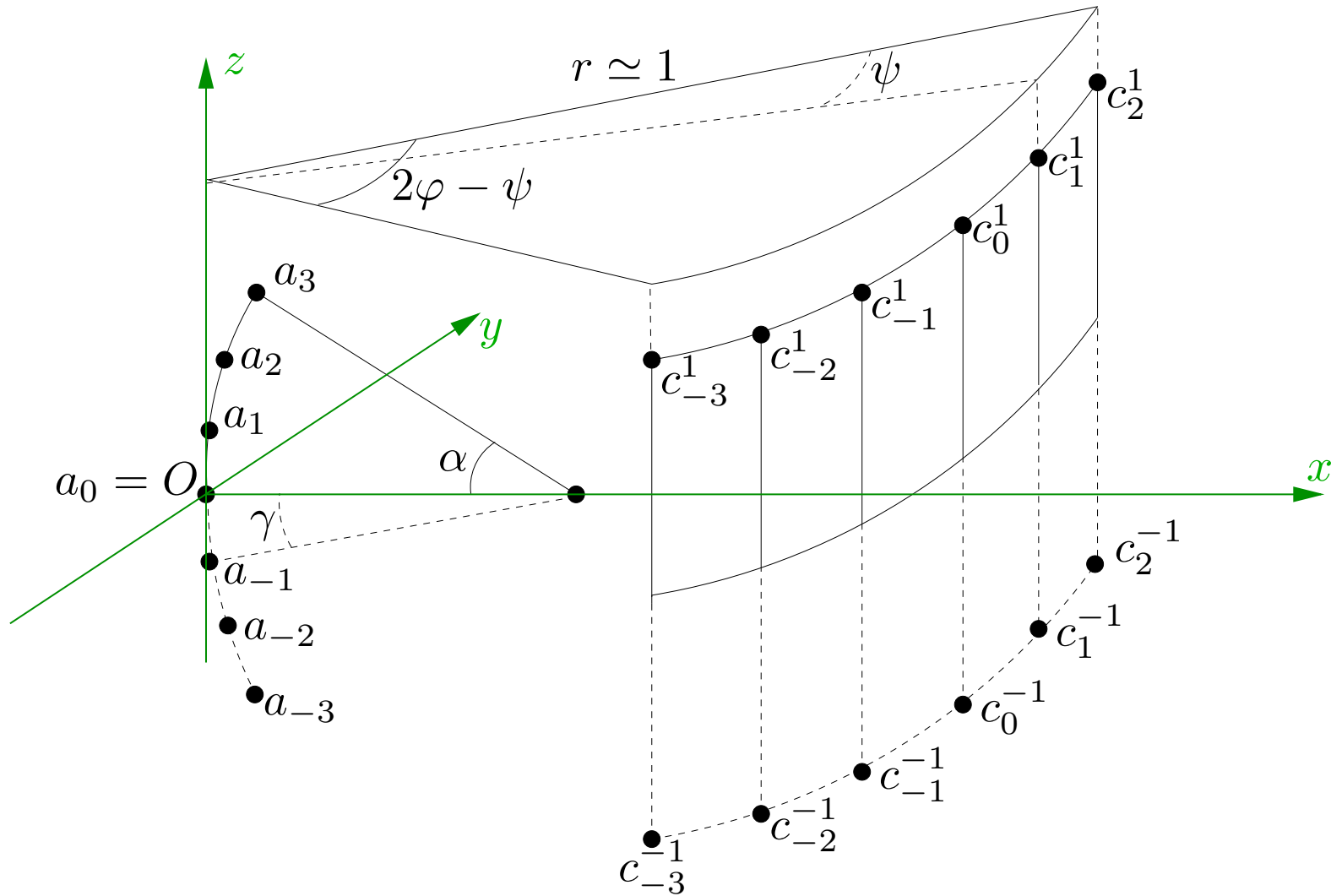
- Example where $n = 3$.



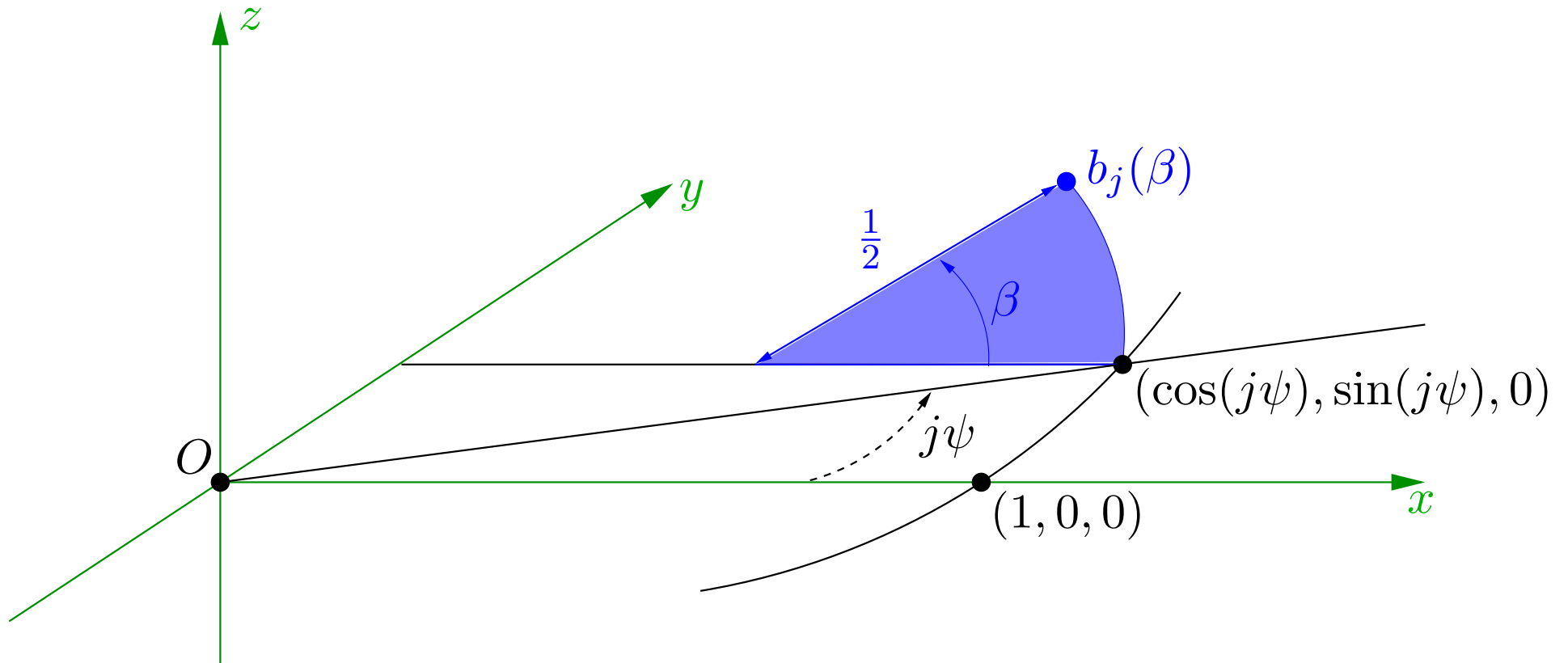
Notation

- $\bar{a} := (a_{-n}, a_{-n+1}, \dots, a_n)$.
- $A := \{a_{-n}, a_{-n+1}, \dots, a_n\}$.
- $\bar{\beta} := (\beta_{-n+1}, \dots, \beta_{n-1})$.
- $B(\bar{\beta}) := \{b_{-n+1}(\beta_{-n+1}), \dots, b_{n-1}(\beta_{n-1})\}$.
- $\bar{b}(\bar{\beta}) := (b_{-n+1}(\beta_{-n+1}), \dots, b_{n-1}(\beta_{n-1}))$.
- $\bar{c} := (c_{-n}^{-1}, c_{-n+1}^{-1}, \dots, c_{n-1}^{-1}, c_{-n}^1, c_{-n+1}^1, \dots, c_{n-1}^1)$.
- $C := \{c_{-n}^{-1}, c_{-n+1}^{-1}, \dots, c_{n-1}^{-1}, c_{-n}^1, c_{-n+1}^1, \dots, c_{n-1}^1\}$.
- $P(\bar{\beta}) := \text{CH}(A \cup B(\bar{\beta}) \cup C)$.

Point sets A and C



Point $b_j(\beta_j)$



- The blue region is parallel to Oxz .
- $\beta \in [-\alpha, \alpha]$

Coordinates of points in A , $B(\bar{\beta})$ and C



$$a_i := \begin{pmatrix} \frac{1}{2}(1 - \cos(i\gamma)) \\ 0 \\ \frac{1}{2} \sin(i\gamma) \end{pmatrix}$$

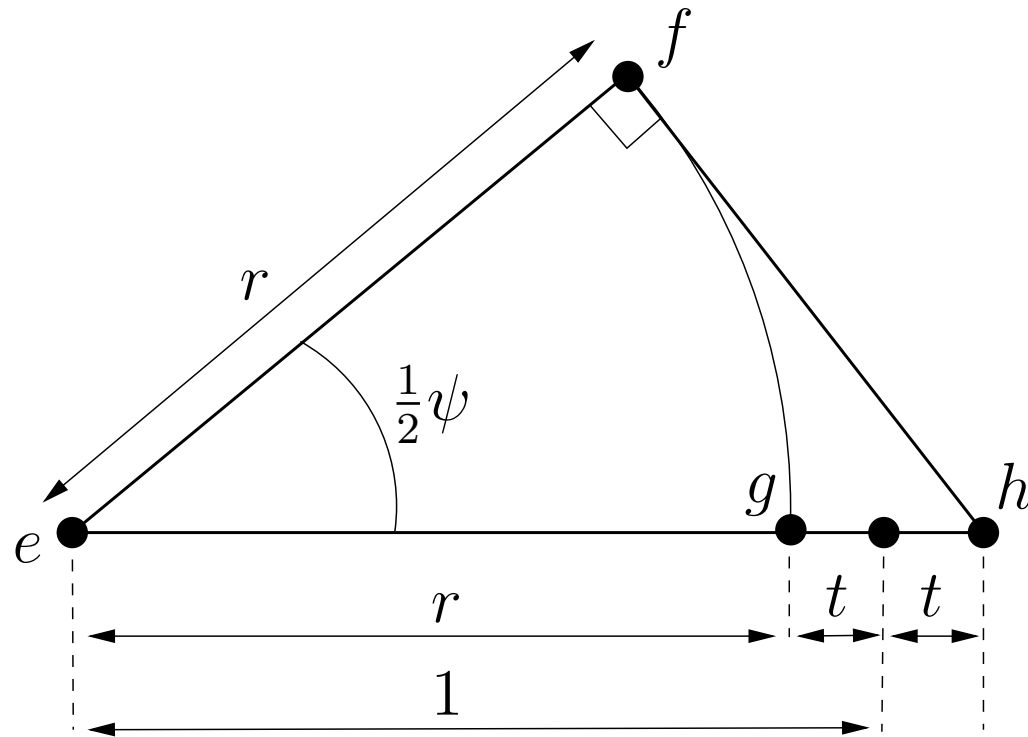


$$c_i^s := \begin{pmatrix} r \cos \left(\left(i + \frac{1}{2} \right) \psi \right) \\ r \sin \left(\left(i + \frac{1}{2} \right) \psi \right) \\ \frac{1}{2} s \alpha \end{pmatrix}$$



$$b_j(\beta) := \begin{pmatrix} \cos(j\psi) - \frac{1}{2}(1 - \cos \beta) \\ \sin(j\psi) \\ \frac{1}{2} \sin(\beta) \end{pmatrix}$$

Parameters



- $\varphi = 1/4n$
- α is small.
- $\psi = \varphi/n$
- $\gamma = \alpha/n$

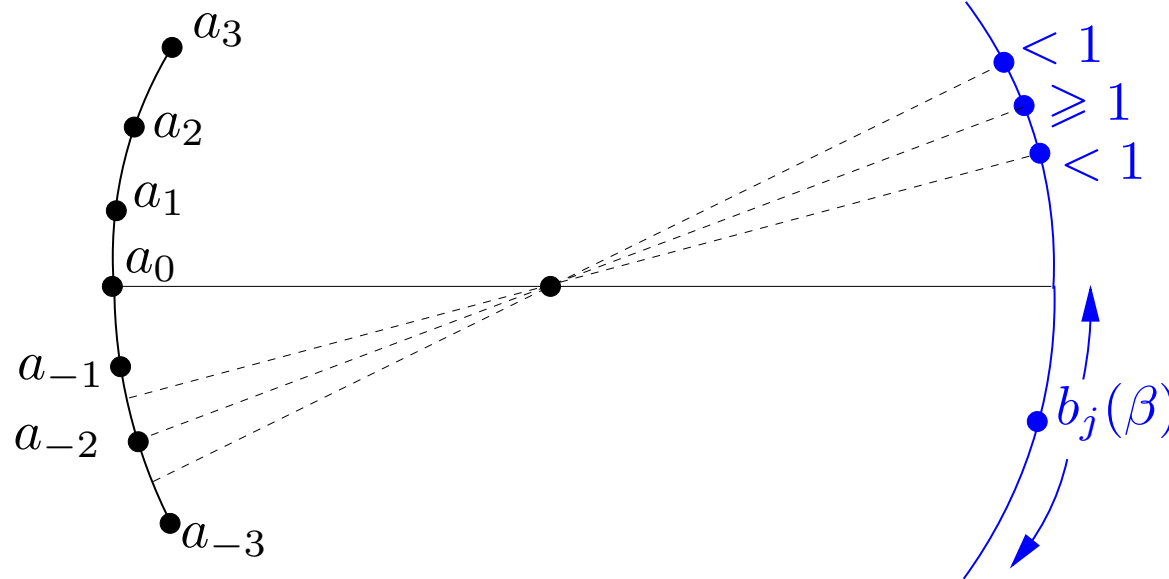
Proof

- Notation: $\text{diam}(E, F) := \max\{d(e, f) \mid (e, f) \in E \times F\}$.

Lemma 1 *The set*

$$\{b_j(\beta) \mid \beta \in [-\alpha, \alpha] \text{ and } \text{diam}(A, \{b_j(\beta)\}) < 1\}$$

has at least $2n$ connected components.



Proof

- Proof of Lemma 1: Calculations, until the second-order terms.

Lemma 2 *The combinatorial structure of $\text{CH}(A \cup B(\bar{\beta}) \cup C)$ is independent of $\bar{\beta}$.*

- We denote $P(\bar{\beta}) = \text{CH}(A \cup B(\bar{\beta}) \cup C)$.

Lemma 3 $\text{diam}(A \cup B(\bar{\beta}) \cup C) = \text{diam}(A, B(\bar{\beta}))$.

Proof

- Definitions:

$$\mathcal{S}_n = \{(\bar{a}, \bar{b}(\bar{\beta}), \bar{c}) \mid \bar{\beta} \in [-\alpha, \alpha]^{2n-1}\}$$

$$\mathcal{E}_n = \{(\bar{a}, \bar{b}(\bar{\beta}), \bar{c}) \mid \bar{\beta} \in [-\alpha, \alpha]^{2n-1} \text{ and } \text{diam}(P(\bar{\beta})) < 1\}$$

- Notice that $\mathcal{E}_n \subset \mathcal{S}_n \subset \mathbb{R}^{24n}$.
- **Lemma 4** *The set \mathcal{S}_n can be decided by an ACT with depth $O(n)$.*
- **Lemma 5** *Any ACT that decides \mathcal{E}_n has depth $\Omega(n \log n)$.*

Proof: By lemmas 1 and 3, \mathcal{E}_n has at least $(2n)^{2n-1}$ connected components. Apply Ben-Or's bound (Theorem 1).

End of the proof

Theorem 2 *Assume that an algebraic computation tree T_n decides whether the diameter of a 3-polytope is smaller than 1. Then T_n has depth $\Omega(n \log n)$.*

- Proof: Take as input graph of T_n the the graph of $P(\bar{\beta})$. Denote by U_n the ACT of Lemma 4. Plug U_n into the accepting leaves of T_{8n} . It gives an ACT with depth the depth of $T_{8n} + O(n)$ that decides \mathcal{E}_n . Apply Lemma 5.

Related work

- (Chazelle) The convex hull of two 3-polytopes can be computed in linear time.
- (Chazelle et al.) It is not known whether the convex hull of a subset of the vertices of a 3-polytope can be computed in linear time.
- (Chazelle et al.) However, we can compute in linear time the Delaunay triangulation of a subset of the vertices of a Delaunay triangulation.

Diameter is harder than Hopcroft's problem

Hopcroft's problem

- P is a set of n points in \mathbb{R}^2 .
- L is a set of n lines in \mathbb{R}^2 .
- Problem: decide whether

$$\exists(p, \ell) \in P \times L : p \in \ell.$$

- An $o(n^{4/3} \log n)$ algorithm is known. (Matoušek).
- No $o(n^{4/3})$ algorithm is known.
- We give a linear-time reduction from Hopcroft's problem to the diameter problem in \mathbb{R}^7 .
- We first give a reduction to the *red-blue diameter* problem in \mathbb{R}^6 : compute $\text{diam}(E, F)$ when E and F are n -point sets in \mathbb{R}^6 .

Proof

- $\theta(x, y, z) := \frac{1}{x^2 + y^2 + z^2} (x^2, y^2, y^2, \sqrt{2}xy, \sqrt{2}yz, \sqrt{2}zx)$.
- Note that $\|\theta(x, y, z)\| = 1$.
- For $1 \leq i \leq n$
 - $p_i = (x_i, y_i, 1)$
 - $\ell_i = (u_i, v_i, w_i)$ is the line $\ell_i : u_i x + v_i y + w_i = 0$.
- Let $p'_i := \theta(p_i)$ and $\ell'_j = \theta(\ell_j)$.
- We get

$$\begin{aligned} \|p'_i - \ell'_j\|^2 &= \|p'_i\|^2 + \|\ell'_j\|^2 - 2 \langle p'_i, \ell'_j \rangle \\ &= 2 - 2 \frac{\langle p_i, \ell_j \rangle^2}{\|p_i\|^2 \|\ell_j\|^2} \end{aligned}$$

Proof

- Note that $p_i \in \ell_j$ iff $\langle p_i, \ell_j \rangle = 0$.
- Thus, there exists i, j such that $p_i \in \ell_j$ if and only if $\text{diam}(\theta(P), \theta(L)) = 2$.
- $\theta(P)$ and $\theta(L)$ are n -point sets in \mathbb{R}^6 .
- Similarly, we can get a reduction from Hopcroft's problem to the diameter problem in \mathbb{R}^7 , using this linearization:

$$\theta(x, y, z) := \frac{1}{x^2 + y^2 + z^2} (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}yz, \sqrt{2}zx)$$

Related work

- The red-blue diameter in \mathbb{R}^4 can be computed in $O(n^{4/3} \text{polylog } n)$ (Matoušek and Scharzkopf). It would be interesting to get a reduction from Hopcroft's problem.
- Erickson gave reduction from Hopcroft problem to other computational geometry problems.